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THE DIAMETER OF THE ANDROMEDA NEBULA

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During recent summers, beginning in 1930, the brightness and colors of several scores of extra-galactic nebulae have been measured with a photoelectric photometer attached to the large reflectors at Mount Wilson. One of the conclusions resulting from this study is that the diameters of a number of these objects are greater than those seen at first sight on photographs. This extension of size is not surprising, as usually each nebula fades from a relatively bright center gradually outward until the outline is lost in the foreground of the sky. Any increase in the known size of extra-galactic nebulae is of course important in a comparison with the dimensions of our own galaxy.

When the photoelectric amplifier using the FP-54 tube was perfected at Madison, our attention was directed to the possibility of detecting light in the outer regions of the Andromeda nebula, M 31, which should be a favorable object for such a test. Trial measures were made in the spring of 1933 with the amplifier on the 15-inch refractor, but it was soon decided to wait until the summer, when the outfit was to be used with the 100-inch reflector at Mount Wilson. The details of the installation on the 100-inch will be published elsewhere. For measures of the Andromeda nebula a focal diaphragm of 8 mm. diameter was ordinarily employed. At 16″.0 per mm. the field was 128″ across and 12,870 square seconds in area. This amount of sky, condensed on the Kunz potassium cell, gave on the average a total light equivalent to a twelfth-magnitude star and could be measured satisfactorily. The telescope was set on a known star, or on the nucleus of the nebula, and measures of the sky, or sky plus nebula, were then taken on the same hour circle at intervals of 10′ in declination. At each pointing the observer at the telescope would examine the field for stars and, if necessary, turn the slide motion in right ascension, sometimes as much as several minutes of arc, until a blank place in the sky was found.

A sample run on the hour circle through the nucleus is given in table 1.

Owing to the diurnal motion, the sky decreased progressively in intensity as the altitude became higher, but a linear change in the sky brightness is eliminated by taking means, forward and back. The net deflections for the nebula given in the seventh column were obtained by subtracting the tentative value 27.4 mm. for the sky from the mean deflections. These differences were then multiplied by the factor 1.26 to reduce to

TABLE 1

SEPTEMBER 15, 1933. SAMPLE RUN NORTH OF NUCLEUS. SCALE AT 1.7 M. SENSITIVITY 35,000 MM./VOLT. 1 MM. = 25.75 MAG./SQ. SEC. READINGS, 20 SEC. SIDEREAL TIME, 20^h7 TO 21^h5.

	DEFLECTIONS			GENERAL				
DECLINATION	FORWARD	MEAN	BACK	MEAN	MEAN	NEBULA	× 1.26	ADJUSTED
	36.5		32.0					
41° 27′	36.0	36.2	31.5	31.8	34.0	6.6	8.3	8.1
	31.2		29.0					
41° 37′	29.8	30.5	28.2	28.6	29.6	2.2	2.8	2.6
	29.5		28.2					
41° 47′	30.2	29.8	28.0	28.1	29.0	1.6	2.0	1.8
	29.0		28.6					
41° 57′	29.3	29.2	28.2	28.4	28.8	1.4	1.8	1.6
	28.5		28.5					
42° 7′	28.8	28.6	29.0	28.8	28.7	1.3	1.6	1.4
	28.2		28.5					
42° 17′	29.2	28.7	27.2	27.8	28.2	0.8	1.0	0.8
	29.0		27.8					
42° 27′	27.5	28.2	28.0	27.9	28.0	0.6	0.8	0.6
	28.0		27.5					
42° 37′	29.0	28.5	28.5	28.0	28.2	0.8	1.0	0.8
	26.8		27.2					
42° 47′	27.8	27.3	27.5	27.4	27.4	0.0	0.0	−0.2
	27.5		28.0					
43° 17′	27.0	27.2	26.8	27.4	27.3	−0.1	−0.1	−0.3

standard sensitivity, 1 mm. = 26.0 mag./sq. sec., and in the last column are the values for the nebula after an adjustment of −0.2 mm. to agree with the other nights. The results of this run, and others on the hour circle of the nucleus taken on three nights, are shown graphically in plate 1.

The observations of the nebula were always connected with one or more comparison stars, and the measures were reduced to the standard sensi-

tivity of 1 mm. = 26.0 mag./sq. sec., for convenient comparison of runs on different nights. The values ranged from 25.72 to 26.13, depending upon variations in the amplifier, due principally to a temperature coefficient of the cell and to small changes in the applied voltage. The deflection of 27.4 mm. for the sky corresponds to 22.2 mag./sq. sec., which is not far from the average for the season.

In the upper part of plate 1, smooth curves have been drawn arbitrarily to indicate the course of the surface brightness, and as the light fades away the evidence of its presence becomes relatively weaker. Points on the curves have been marked at each whole magnitude, and we may characterize the different steps somewhat as follows:

24 mag./sq. sec.—Conspicuous
25 mag./sq. sec.—Certain
26 mag./sq. sec.—Probable
27 mag./sq. sec.—Suspected.

With sufficient optimism the relative degrees of certainty might be pushed outward by one step each, but we prefer to keep on the safe side.

These magnitudes, charted in the lower part of the plate, which is from Ritchey's well-known negative taken at the Yerkes Observatory, indicate that the nebula extends far beyond the limits of the photograph. Measures at other positions around the nebula were also made during the season as opportunity offered, and the inferred intensities are placed on the chart. In some of the exploratory runs the extension is not so well determined as that north and south of the nucleus, but there was always evidence for outlying illumination. The regions for 25 mag./sq. sec. are marked by small circles of the size of the diaphragm used in the measures. The large oval indicated by the broken line is merely a suggestion of the possible outline of the nebula for this degree of intensity. There seems to be an abrupt change not far from 25 mag./sq. sec., as though the main nebulosity stopped at that limit, but such a conclusion requires confirmation. Two lines on the chart indicate regions which were found incidentally to have intensities of the order of 23 and 24 mag./sq. sec., respectively.

From the measures on the hour circle through the nucleus the residuals of 214 deflections for 107 means give ± 0.45 mm. for the probable error of one deflection, or ± 0.23 mm. for the mean of four for one observed point. As the average deviation of one of these points from the smooth curve is ± 0.36 mm. there has been no forcing of the data beyond what is warranted by the internal agreement of the measures. In fact, were it not for the interference of the field stars, the detection with the amplifier of 0.40 mm. deflection, corresponding to 27 mag./sq. sec., would really be easy.

It is difficult to estimate the extension of the hitherto accepted diameter

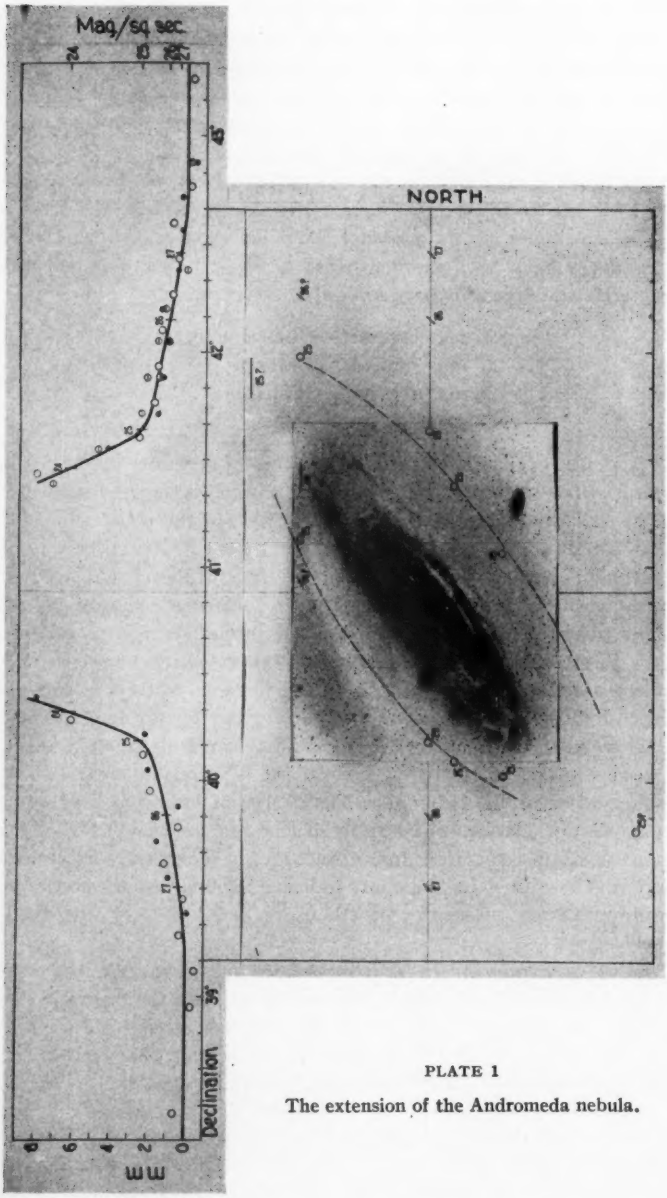

PLATE 1

The extension of the Andromeda nebula.

of the nebula given by the present work. The outline for 25 mag./sq. sec. is roughly 50′ wide, whereas the photographic dimensions adopted by Hubble are 40′ × 160′; thus the factor of increase is about 1.25. The extremes of 26 and 27 mag./sq. sec. on the hour circle of the nucleus give corresponding factors of 2.2 and 2.8, respectively. If the 160′ for the major axis of the nebula is multiplied by the last factor we have 450′, which is of the same order as the 7° suggested by Hubble as the maximum diameter indicated by several outlying objects, assumed to be globular clusters associated with the nebula. It is by no means certain, however, that the nebulosity which we measure at 26 or 27 mag./sq. sec. is as much elongated in outline as the brighter portions. It is quite possible that the isophotes for the outer regions become more nearly circular. The tests for extension of the nebula along the major axis will require considerable work during another season.

From the present data we conclude that the known diameter of the Andromeda nebula has been more than doubled in the direction north and south from the nucleus, and that this same ratio of increase applies to the apparent minor axis or width of the nebula.

Our observations were primarily directed to the outer regions of the nebula, but in the course of the work a few measures were also made near the nucleus. For an object with a spectrum near the solar type like M 31, the photoelectric magnitude is 0.1 or 0.2 mag. brighter than the photographic; but for rough purposes the two scales may be considered as the same. We have measured a circle of 1′ radius about the nucleus to be equivalent to magnitude 7.9, which is not far from Seares's[1] estimate of 7.8, photographic. We also determined the intensity at 1′ from the nucleus to be 18.6 mag./sq. sec. as compared with his 18. Seares[2] places the apparent photographic surface brightness of our galaxy in the neighborhood of the sun, seen from outside and normal to the galactic plane, at 23.7 mag./sq. sec. We therefore confirm his conclusion that the central region of the Andromeda nebula is at least 5 magnitudes or 100 times brighter than our part of the galaxy. On plate 1 the surface brightness 23.7 is barely visible. Hubble[3] has found that the limiting photographic magnitude distinguishable from the sky foreground is 25.2 mag./sq. sec. for faint images of extended objects. At present it is probably quite out of the question to confirm by photography the extension of so large an object as the Andromeda nebula out to regions where the intensity is only one or two per cent of the surface brightness of the sky.

Adopting +5.3 as the photoelectric absolute magnitude of the sun and $m - M = 22.0$, corresponding to a distance of 250,000 parsecs, as the modulus for the nebula, we find that the light in the measured small circle at 25 mag./sq. sec. is equivalent to absolute magnitude −7.3, or to 110,000 suns. For 27 mag./sq. sec. the number is 17,000, while at

the nucleus the light from the same area is equivalent to 75,000,000 suns.

The extension of the Andromeda nebula beyond the photographic limits is presumably typical of what may be found for other such objects. Comparisons with the surface brightness of the galaxy are still hampered by uncertainties in the data for our own system. There may be no part of the galaxy which, seen from outside, would be as bright as the central portion of M 31. In the latter system the region comparable with the sun's neighborhood may well be 1° out from the nucleus, or at a distance of say 4000 parsecs, where, according to the data shown on plate 1, the surface brightness agrees fairly well with Seares's previously mentioned estimate for our own system. A previous suggestion[4] gave 30,000 parsecs for the diameter of the galaxy, with the sun 10,000 parsecs from the center, as against 10,000 parsecs for the diameter of the nebula. The present work indicates that the diameter of the nebula may well be as much as 20,000 parsecs. The figure for the galaxy may have to be revised still further when the space absorption near the median plane is better determined. In fact, when the size of our galaxy is as well known as that of the Andromeda nebula, most of the inferred difference in scale between the two systems may disappear.

The construction of the amplifier at Madison and its use at Mount Wilson were made possible by grants from the National Research Council and from the California Institute of Technology, the latter in connection with plans for the 200-inch reflector.

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[1] *Mt. Wilson Contr.*, No. 191, p. 12; *Astrophys. J.*, Chicago, Ill., **52**, 162–182, 177 (1920).

[2] *Ibid.*, p. 5.

[3] *Mt. Wilson Contr.*, No. 453, p. 11; *Astrophys. J.*, **76**, 106–116 (1932).

[4] Stebbins, J., these PROCEEDINGS, **19**, 227 (1933).

HAPLOIDS IN HORDEUM VULGARE

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Reports of the occurrence of haploids in various plants have been becoming increasingly more frequent, so that interest now centers perhaps not so much in such accounts *per se* as in the genus or species in which the phenomenon is found to occur.

In the Gramineae the only reported haploid is apparently the one described by Gaines and Aase[1] as having arisen among some crosses between

Triticum compactum humboldtii (*n*-21) and *Aegilops cylindrica* (*n*-14). This particular haploid possessed the same number of chromosomes (2*n*-21) as the *Triticum* parent, consequently it was classified as a *Triticum* haploid.

While preserving root tips of *Hordeum vulgare*, it was observed that the seedlings were a diverse lot. Most of them were large and robust with plenty of suitable root tips, while a good proportion of the remainder were small and puny with the first leaves barely emerging, decidedly lacking in chlorophyll, and from such seedlings not more than two or three root tips were obtainable. The seeds from which these abnormal seedlings arose appeared to be both thicker and longer, in the sense of having more copious endosperm, than the seeds attached to the normal seedlings. The differences were thought to be due to poor germination as the seeds were somewhat old. The seedlings were discarded after removal of the root tips.

FIGURE 1

Typical metaphase plate from a normal diploid variety of *Hordeum vulgare*.

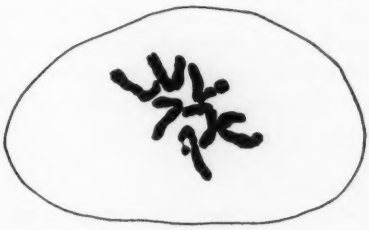

FIGURE 2

Typical metaphase plate from the root tip of a haploid plant. × 1465.

When the preparations were checked over it was immediately noticed that an occasional root tip appeared to have far fewer chromosomes than usual. The variety was presumed to be a representative diploid one with 14 somatic chromosomes. The metaphase plates are so clear that any with a few chromosomes more than or less than the regular number are readily detected. Critical examination revealed the probable presence of haploidy.

The seeds, presumably of the 1931 crop, were purchased in 1932 from the Ferry-Morse Seed Company of San Francisco and germinated in November, 1933. Nothing is known of the history of the seeds hence we are left to assume that the haploids arose spontaneously. The mystery is how such an apparently high proportion of haploids could have arisen in a commercial field crop.

The cytology of *Hordeum vulgare* L. has already been adequately discussed,[2] consequently a brief description of the morphology of the somatic

chromosomes of a normal diploid from the same lot of seeds from which the haploids came, is all that seems to be required.

With the exception of two elements, the somatic chromosomes in the diploid roots are essentially alike, as is shown in figure 1. The primary constrictions are submedian on all chromosomes. The two exceptional chromosomes each possess a secondary terminal constriction. This constriction is usually shallow, occasionally rather deep; very rarely the constricted segment may become separated from the main body by a short thick thread. It may be mentioned that inconclusive evidence of the presence of similar terminal constrictions on at least two other chromosomes has been noted. There appear to be no trabants in most of the diploid roots, as in the one from which figure 1 was taken, but in a very few there were two chromosomes each with a single inconspicuous trabant.

Of the seven chromosomes in the somatic cells of the haploid plants, five are alike, with submedian primary constrictions as in normal diploids, one chromosome has a detached terminal segment in addition to the primary constriction, and another submedianly constricted chromosome possesses a medium sized to large trabant at the end of a long slender thread, as is illustrated in figure 2. The detached portion on the segmented chromosome appears generally to have a greater diameter than the short arm to which it is attached. The chromosomes in the haploid plants are more sharply bent at the primary constriction than are those in normal diploids.

It is, of course, impossible even to venture an opinion regarding the origin of the haploids, particularly in view of the fact that in the batch of seedlings mentioned above the dwarf seedlings with very few root tips constituted approximately ten per cent of the total number. Out of the first twelve slides (each slide contained three root tips sectioned simultaneously) four each held a single haploid root tip. The apparent "mass production" of haploids in a field culture seemingly does not permit the application of any of the proved explanations for the origin of other known haploid plants.

The writer is greatly obliged to Dr. J. Clausen for the use of his superb microscopical equipment and for generously assisting in checking over the literature.

[1] Gaines, E. F., and Aase, H. C., *Amer. Jour. Bot.*, **13**, 373–385 (1926).

[2] For the pertinent literature, consult Ghimpu, V., *Arch. d'Anat. Micros.*, **26**, 135–250 (1930).

POSSIBLE CYTOPLASMIC AS WELL AS CHROMOSOMAL CONTROL OF SEX IN HAPLOID MALES

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In an earlier discussion[1] of the genetics of haploid males among animals, I called attention to the fact that such individuals are only *somatically* male, since the single type of sperm which they produce is female determinative when it functions in fertilization. I then wrote: "What makes these haploid individuals somatically male (producers of sperm rather than eggs) we are at present unable to state." This inability has perhaps been removed by subsequent discoveries to which I wish to call attention.

It has long been known that the egg of the honey-bee, if it develops without fertilization, becomes a male. Yet the sperm produced by such a male, if it fertilizes eggs, converts them all into females. It would seem, then, that in such species the haploid state leads to sperm production (somatic maleness), the diploid state to egg production (somatic femaleness). But all gametes in such species are genetically female, since that is the sex of all zygotes which they produce.

On the other hand, Bridges[2] has shown that in Drosophila the haploid tissue of mosaics is somatically female, which is contrary to the rule for haploids in insects such as the honey-bee. How are we to reconcile these discordant facts? Bridges' theory of genic balance seems to fit all categories of sex determination except this one. Each chromosome, according to Bridges, has a definite sex tendency due to the collective action of its genes. The X-chromosome of Drosophila has a net female tendency superior to the male tendency of the autosomes. Bridges rates them, respectively, as $X = +100$, $A = -80$. Accordingly, the combination $2X2A$ is a female $(+40)$, its sign being positive, but $X2A$ is a male (-60). Bridges takes into consideration only the chromosomes as sex-determinating influences in the gametes, but if, following Goldschmidt, Correns and Harder, we ascribe such an influence to the cytoplasm as well as to the chromosomes, the difficulty vanishes. We have only to suppose that the sex tendency of the cytoplasm, in insects such as the bee, is strongly male, sufficiently so to overbalance the female tendency of a *single* set of chromosomes, yet inferior to the female tendency of *two* sets of chromosomes. Thus the haploid will be *somatically* a male, but its gametes will be female determinative when they function in fertilization, since they contain a full set of chromosomes identical in character with those of the egg, and two such sets produce a diploid female.

Whiting[3] has shown that in the parasitic wasp, Habrobracon, females

are diploid and males ordinarily haploid, as in other Hymenoptera. Yet under inbreeding, a few diploid males may be produced. By very convincing genetic evidence he has shown that the females are sex heterozygotes, as are females in moths and birds. Their formula is given as $XY + 2A$. The haploid males are of two sorts such as might arise from unfertilized but matured eggs, viz., either $X+A$ or $Y+A$. Such also is the composition of the sperm which they respectively produce. When an X egg is fertilized by Y sperm or a Y egg by X sperm, females result, $XY+2A$ individuals. If an X egg is fertilized by X sperm, or a Y egg by Y sperm, diploid males result either $XX+2A$ or $YY+2A$ in chromosome constitution. This unparalleled situation, however, falls in line with the theory of balance in sex determination, if we admit the cytoplasm as well as the chromosomes to a share in it. We need only suppose that in this case the cytoplasmic influence—male in tendency, as in other Hymenoptera—is strong enough to overbalance the female tendency of the chromosomes in any combination either haploid or diploid, except one into which *dissimilar sex chromosomes* enter. This heterozygote has sufficient "hybrid vigor" to allow the female sex tendency to come to expression and eggs rather than sperm to be produced.

A like superiority of heterozygous over homozygous combinations is effective in the case of self-sterile plants (East, 1933).[4] Homozygotes for self-sterility factors are secured only with great difficulty, so much slower do the pollen-tubes grow in homozygous combinations than in heterozygous ones. It is significant that Whiting reports the production of diploid males only when the uniting gametes are from closely related stock, a condition under which heterosis would be at a minimum. He also notes simultaneously the occurrence of any eggs which fail to develop, another indication that the homozygous combinations are either inviable or develop only with difficulty. This suggests that union of gametes alike in chromosome constitution occurs only when dissimilar ones are not available, which is substantially the situation in self-sterile plants.

Summary.—Haploid males are only somatically male. The chromosome complexes which they transmit are potentially female determinative. Bridges' theory of balance in sex determination will account for this and similar cases, if it is assumed that cytoplasm as well as chromosomes may influence somatic sex determination. Heterosis is the decisive factor in diploid sex determination in Habrobracon, as shown by Whiting.

[1] *Proc. Nat. Acad. Sci.*, **16**, 783–791, 12 (1930).

[2] *Sex and Internal Secretions*, Williams and Wilkins Co., Baltimore, p. 55 (1933)

[3] *Science*, **78**, 537, Dec. 8 (1933).

[4] *Proc. Nat. Acad. Sci.*, **19**, 841–845 (1933).

MUTATION RATE INCREASED BY AGING SEEDS AS SHOWN BY POLLEN ABORTION

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Navashin, in March, 1933, reported extremely high numbers of chromosomal aberrations in roots of *Crepis tectorum* plants grown from six and seven year old seeds.[1] The changes reported could not be studied genetically. They were chiefly translocations, and were detected in 22, or 81 per cent, of the 27 plants studied, but were found in only 0.1 per cent of the plants grown from fresh seeds. As was also the case when similar types of abnormalities were obtained by treatment with heavy dosages of x-rays, those from aged seeds occurred in sectors of the plants. It was realized that the technique used in studying pollen abortion in Datura might afford a very favorable means of confirming Navashin's findings and possibly of extending them to include gene mutations, as well as to study the inheritance of the changes brought about by this means.

In Datura two chromosomal mutations and one gene mutation affecting pollen abortion have been found in nature. The pollen abortion gene has been located in a particular chromosome by breeding tests. Mutations of both types are readily induced by radiation treatments, especially by x-raying seeds with heavy dosages. The mutations induced usually occur in limited sectors of most of the plants grown from the treated seeds. In some cases single plants have shown several types of pollen abortion due to as many mutations. We are able, in most cases, to distinguish between chromosomal and gene mutations affecting pollen abortion. The chromosomal type shows aborted grains that are small, shriveled and empty. These occur in rather definite proportions, and in many cases include about 50 per cent of the grains. The grains which are aborted because they contain a pollen abortion gene regularly include 50 per cent of the total number. They are marked by sub-normal to very small size, and usually by differences, which are interpreted as degenerative changes, in the character of their contents. Pollen abortion of the order found in plants with extra chromosomes (trisomics) or plants with extra sets of chromosomes (euploid series),[2] is not to be confused with the types just described. While it involves shriveling of the grains as in the case of chromosomal mutations here considered, the percentage of aborted grains in trisomics is generally lower and more variable. Moreover, the presence of an extra chromosome is nearly always detected by the altered appearance of the plant carrying it. No trisomics nor euploid

types were recorded among the plants used in the experiments to be described.

Pollen abortion affords at once a convenient and delicate index to the mutation rate. About half of the known chromosomal mutations, other than trisomic and euploid types, and a very common sort of gene mutation are indicated by this character. The need for cytological preparations may be largely eliminated in the finding of chromosomal mutations. Since the pollen abortion genes are recognizable in the plants grown directly from the treated seeds, the culture of a second generation with thirty or more times as many individuals, as would be required to disclose a recessive visible gene, is unnecessary. Both sorts of pollen abortion mutations, may, in most cases, be transmitted to the offspring, although pollen abortion genes are transmitted only through the female gametes. Since the mutations show in the flowers, cross- or self-pollinations may be made on the individual flowers from which pollen samples have been recorded.

The standard Line 1 of Datura is highly inbred, and also has been passed through a haploid parent. Seeds from this line which had been aged from four to ten years by dry storage at room temperature yielded 628 recordable plants in 41 individual progenies. Two flowers from each of these plants were examined for pollen abortion mutations, and 19 cases of the gene type and 26 cases of the chromosomal mutation type were found. Most of the pollen abortion mutations were restricted to sectors composing one-half or smaller parts of the plants, but in seven cases the whole plant apparently was involved. A second series of plants grown from Line 1 seeds aged from one to five years was similarly tested. One of the 26 parent plants of this series produced a progeny which was discarded because it segregated for a pollen abortion gene. The mutation probably occurred in a sector of the parent plant, and had not hitherto been noticed either in the parent or in its offspring. The other progenies contained 2174 recordable plants, and in this series 29 gene type and 31 chromosomal type pollen abortion mutations were found. Ten of these mutations apparently involved the whole plant, and the others were limited to sectors of half or less than half of a plant. As a control, two flowers from each of 331 Line 1 plants, grown from seeds less than one year old, were examined for pollen abortion mutations. One each of gene and chromosomal type mutations were found in this control series. The mutations found in the two series and their controls are tabulated according to the age of seed, under "two-flower samples," in table 1. The four to five year old seed group in this table is composed of plants from both series. The older seed groups summarize the remainder of the first series, and the younger seed groups summarize the remainder of the second series and the controls. The percentage of mutations found for each age group

(table 1) seems to be roughly proportional to the age of the seeds tested. Although the nine to ten year old seed group shows a somewhat lower percentage of mutants than the seven to eight year old group, the other groups show percentages of mutations that decrease with decreasing age of seeds.

Through the kindness of Miss S. Satin, cytological observations have been made on 10 of the gene type and on 11 of the chromosomal type of pollen abortion mutations. The records (table 2) show that the gene types had the expected normal chromosomes. Chromosomal abnormalities, as listed in table 2, were demonstrated in eight of the chromosomal mutation types. That the other three showed no chromosomal abnormality may be due either to error in predicting the type of mutation causing the pollen abortion, or to failure to delimit the mutant sector

TABLE 1

POLLEN ABORTION MUTATIONS AND AGE OF SEED

AGE OF SEED IN YEARS	ONE-FLOWER SAMPLES			TWO-FLOWER SAMPLES				
				POLLEN ABORTION TYPES				
	PLANTS RECORDED	TOTAL TYPES	PERCENTAGE OF TYPES	PLANTS RECORDED	GENES	CHROMOSOMAL	TOTAL	PERCENTAGE OF TYPES
9–10	100	7	7.0 ± 1.7	100	4	3	7	7.0 ± 1.7
7–8	242	16	6.6 ± 1.1	242	6	15	21	8.7 ± 1.2
6–7	14	0	.....	14	0	1	1	7.1 ± 4.6
5–6	49	2	4.1 ± 1.9	49	1	2	3	6.1 ± 2.3
4–5	604	20	3.3 ± 0.5	604	18	13	31	5.1 ± 0.6
3–4	1289	19	1.5 ± 0.2	1018	14	15	29	2.8 ± 0.3
2–3	911	6	0.7 ± 0.2	549	3	5	8	1.5 ± 0.4
1–2	1324	11	0.8 ± 0.2	226	1	2	3	1.3 ± 0.5
0–1	939	4	0.4 ± 0.1	331	1	1	2	0.6 ± 0.3
Totals	5472	85		3133	48	57	105	

correctly. Twenty-four normal plants from the same groups showed no chromosomal mutations in the single samples that were cytologically examined. Both chromosomal and gene types are to be considered genetic changes since a high proportion of similar abnormalities found in nature as well as those induced by radiation treatment have been transmissible. A certain number, as is true of some pollen abortion types from radiated seeds, will fail to reach the next generation in cases in which the gene or chromosomal abnormality cannot be carried by the egg cells.[7]

Seeds aged from less than one to four years old furnished an additional series of 2339 diploid plants among the offspring of Line 1 trisomics. These plants were tested for pollen abortion mutations by the examination of one flower from each plant. The 19 mutations found are included, together with those found in the first flower from each plant of the other series, under "one-flower samples" in table 1. It is evident from this

tabulation also that the percentage of mutation depends in general upon the age of the seeds.

An analysis of table 1 indicates clearly that not all of the mutations that must have occurred have been found. Among 3133 plants 66 pollen abortion mutations were found in the first flower examined and 39 others in the second flower. When a mutation type was found, other flowers were collected from the plant in order to establish the validity of the mutation type and to delimit the sector of the plant in which the mutation occurred. The first flower samples disclosed sectorial mutations in 49 plants. The second flower samples, taken at random position on the plant, are expected to disclose about one-half fewer new mutations than the number found in the first flower samples, if the sectors include one-

TABLE 2

CYTOLOGICAL CHECK ON POLLEN ABORTION TYPES FROM OLD SEEDS
(Miss S. Satin's records)

POLLEN ABORTION, GENE TYPE		POLLEN ABORTION, CHROMOSOMAL TYPE	
PLANT NUMBER	CYTOLOGY	PLANT NUMBER	CYTOLOGY
3201 (183)	12 bivalents	320108 (318)	12 bivalents*
320103 (69)	12 bivalents	320132 (15)	deficiency
320121 (64)	12 bivalents	320211 (26)	12 bivalents*
320133 (58)	12 bivalents	320538 (15)	necktie
320538 (5)	12 bivalents	320538 (17)	chain of four
320538 (9)**	12 bivalents	320545 (28)	translocation
320544 (10)	12 bivalents	320547 (1)	necktie; translocation
320557 (17)	12 bivalents	320548 (6)	necktie
320569 (18)	12 bivalents	320548 (8)	12 bivalents*
320571 (6)	12 bivalents	320557 (7)	chain
		320560 (4)	necktie

* Possible errors in prediction, or perhaps due to failure to locate mutant sector exactly.

** This plant, included in table 1, as having a chromosomal type, was later found to have a gene type mutant sector also.

half (Fig. 1*B*) or nearly one-half of the plant. However, the second (or subsequent) flower samples will be about as efficient as the first in locating small sectors. Thus, on the average, three-fourths of the sectors which involve half-plants may be expected to be found by the method of two-flower samples. Similarly, fewer than one-half of the sectors composing quarter plants (Fig. 1*C*), and a smaller fraction of the whole number of smaller sectors (Fig. 1*D*) will be found by the two-flower sample method. Diagrams of individual plants showing pollen abortion mutation sectors of different sizes are shown in figure 1. The examination of extra flowers brought to light in several cases a second mutant sector with a pollen abortion type distinct from that already recorded. Since these were not found in the first two flowers they are not listed in the tables.

The criteria imposed for the listing of a mutation type have been (1) that the type has been found in the first or second flower sample and (2) that the type has been recovered in at least one flower near the one of first record—with the exception that a few cases with 50 per cent aborted pollen characteristic of chromosomal mutation types have been listed on the evidence of a single record.

The percentages of germination in the various progenies tested varied very widely in all age groups of seeds over one year old. On the average the younger seeds showed a higher percentage of germination than the older seeds. If induction of mutations depends, as Heribert Nilsson[3] has suggested, upon the differential viability of seeds and the selective elimination of normals, the percentages of mutations should be inversely proportional to the percentages of germination. In the Datura experiments, however, both those progenies with the highest percentages of

TABLE 3

GERMINATION AND POLLEN ABORTION MUTATIONS FROM OLD SEED

PERCENTAGE GERMINATED	PROGENIES INCLUDED	PLANTS RECORDED	NUMBER OF MUTATIONS	PERCENTAGE OF MUTATIONS
1–10	5	11	0	0.0
11–20	5	19	2	10.5
21–30	2	17	1	5.9
31–40	7	129	7	5.4
41–50	9	216	7	3.2
51–60	5	183	9	4.9
61–70	4	149	11	7.4
71–80	14	840	26	3.1
81–90	7	607	13	2.1
91–100	8	962	29	3.0
Totals	66	3133	105	

mutations, and those with none, are distributed over almost the entire range of germination percentages. In table 3 all of those plants listed under "two-flower samples" in table 1 are tabulated according to germination percentages at class intervals of ten per cent. No direct relation between the percentage of mutation and the percentage of germination appears. For plants in progenies with less than 50 per cent germination the percentage of mutation is 4.3 ± 0.7, while for those in progenies with more than 50 per cent germination it is 3.2 ± 0.2. The difference is 1.1 ± 0.8 per cent. When this is compared with the mutation percentages of 7.9 ± 0.9 for plants from seeds more than five years old, and 2.7 ± 0.2 for plants from seeds less than five years old, with a difference of 5.2 ± 0.9 per cent, it is clear that the age of the seed and not differential viability determines the mutation rate.

Both the accumulation of mutations with the age of the seeds and the occurrence of most mutations in sectors of the plants indicate that

the mutant sectors are descended from initial cells of the embryonic shoot which have mutated within the resting seed. The number of initial cells which normally take part in the formation of the shoot is not known. There must be at least three initial cells concerned in those plants with two mutant sectors and normal tissue also (Fig. 1*E*). Many of the plants from the older seeds showed, particularly in their younger stages, marked abnormalities of growth and development such as failure of leading buds and irregular branching. Some of these plants died while others became more nearly normal in appearance as lateral branches grew out. The shoots which finally developed seemed to have grown from only a few of the initial sectors originally present. If these initial sectors contained pollen abortion mutations then large sectors or the whole of the adult shoot would, presumably, show the mutations. Non-sectorial mutant plants might have resulted from a mutation in one of the gametes of the parent. However, if all were of this origin the rate should be about the same from seeds of all ages. Although the numbers are small the data suggest that those mutations which apparently involve the whole plant (Fig. 1*A*) may also be more frequent from old seeds than from young seeds. The rate is 1.39 ± 0.25 per cent from seeds more than four years old and 0.23 ± 0.07 per cent from seeds less than four years old, with a difference of 1.16 ± 0.26 per cent. Thus a sort of developmental selection may account for the representation from a very few or only one of the shoot initials in the matured plant.

All the seeds used in these experiments were stored in paper envelopes which were placed in open wooden trays on wooden shelves. Some were in the attic, some in the second story of the laboratory building and others in one of the greenhouse headhouses. Radiations from natural sources seem to be insufficient to account for the rate of mutation found. In the basement of the laboratory building there was an x-ray machine which was operated at intervals during part of the time that the seeds were stored. The minimum linear distance from the x-ray machine to the seeds was slightly over 30 feet. The target was shielded with a one-eighth inch lead screen, and the rays were directed downward, i.e., away from the seeds. A rough calculation showed that not more than 25 *r*-units would have reached the seeds during their storage had the radiation been directed toward them. The experimentally effective dosages of x-radiations on dry seeds ranged around 10,000 *r*-units. Some of the three year old seed had been stored in the building with the x-ray machine and others in the headhouse. A series of plants from these two lots of seeds was tested by examination of three flowers each. The position of the flower was not at random, but such that both of the main forks of each plant were sampled. One gene type and five chromosomal types of mutations responsible for pollen abortion were found in the 259 plants

from seeds that had been stored in the second floor laboratory. The percentage of mutation was 2.3 ± 0.6 per cent. The seeds which were stored in the headhouse, that is, apart from the x-ray machine, gave nine gene type and nine chromosomal type mutations, or 4.5 ± 0.7 per cent from 400 plants. The difference in rates of mutation is 2.2 ± 0.9 per cent. It is possible that factors other than the age of seed may be concerned in the increased mutation rate found. The most obvious difference in the conditions of storage of the two lots of seeds was in the humidity of the storage places. The seed stored under the more humid conditions of the headhouse of the greenhouse gave the higher mutation rate.

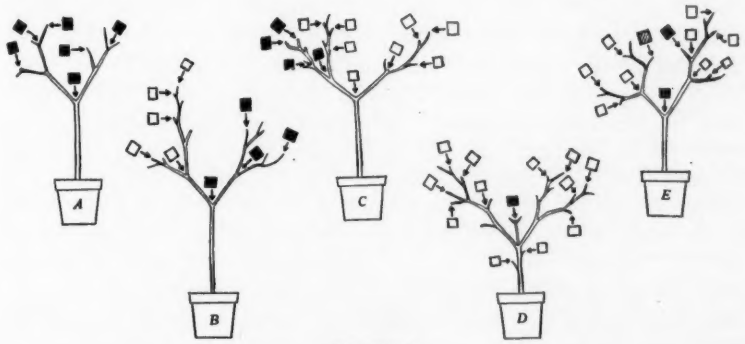

FIGURE 1

Diagrams of plants showing various sizes of mutated sectors characterized by pollen abortion. Arrows from solid blocks indicate positions of flowers which showed pollen abortion. Arrows from outlined blocks indicate positions of flowers with normal pollen. *A* (plant 3201771(49)), sector includes whole plant. *B* (plant 3201769(104)), sector includes one-half plant. *C* (plant 3201780(94)), sector includes one-fourth plant. *D* (plant 3201776(63)), small sector. *E* (plant 3201779(92)), sectors due to two different mutations; one sector (solid blocks) gene type, other sector (shaded block) chromosomal type mutation.

While the old seed experiments in Datura were in progress two papers relating to the problem have appeared. Peto[4] reports changes in roots of corn, similar to those found by Navashin in Crepis, following aging of seeds. He also got like effects from heating seeds of barley. Navashin and Shkvarnikov[5] have also made a preliminary report on increase of mutation rate by heat treatment of dry seeds. Results comparable to those obtained by aging Crepis seeds for six or seven years, or by treating dry seeds with x-ray dosages of 15,000 *r*-units, were secured by heating seeds for 20 days at 54 to 55 degrees centigrade. The percentage of germination obtained from these seeds was relatively high. Navashin has suggested that the chromosomal abnormalities which he has studied in Crepis roots

may be attributed to the metabolic condition of the cells in the dormant embryos.[6]

The mutations that occurred in stored seeds of Datura would seem to have been spontaneous. It is not as yet clear, however, what the ultimate mechanism may be that brings about genetic changes in aged and heated seeds. Further study of the mutation rate in this material requires at least the control of temperature and moisture differences. Seeds stored in the ground under natural conditions are subjected to a number of environmental factors such as temperature, moisture, light, oxygen supply, earth radiations and cosmic rays. So far we have very little knowledge of the effects upon the mutation rate of various combinations of these conditions for varying lengths of time. The experiments on Datura here reported show that the mutation rate, as indicated by pollen abortion, is increased with age when seeds are stored at room temperature. But the conditions to which seeds are exposed in nature are quite different. Seeds of Datura have retained their viability after being buried in the soil for considerable lengths of time. It would be profitable to test plants grown from such seeds in order to see if aging seeds under more natural conditions would have effects similar to those found in our experiments. Excavated soil sometimes gives rise to a crop of Jimson weeds under conditions which indicate that they came from seeds buried for a long time. We should appreciate information regarding any such cases in which it would be possible to subject the material to genetic tests.

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[2] Blakeslee, A. F., and Cartledge, J. L., these PROCEEDINGS, **12**, 315–323 (1926).

[3] Nilsson, N. H., *Hereditas*, **15**, 320–328 (1931).

[4] Peto, F., *Canad. Jour. Res.*, **9**, 261–264 (1933).

[5] Navashin, M., and Shkvarnikov, P., *Nature*, **132**, 482 (1933).

[6] Navashin, M., *Planta, Arch. wiss. Botanik*, **20**, 233–243 (1933).

[7] Both the gene type and the chromosomal type have been recovered in the second generation in seedlings now in the greenhouse. The two types of pollen abortion from aged seeds therefore may be transmitted.

THREE CASES OF DEFICIENCY IN CHROMOSOME 9 OF ZEA MAYS

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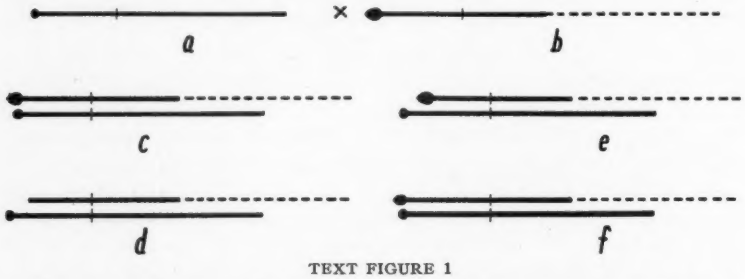
The purpose of the present investigation was to locate by means of x-ray induced deficiencies the regional position of the genes yg_2 (yellow-green), *C* (colored aleurone), *sh* (shrunken endosperm) and *wx* (waxy endosperm) in chromosome 9 of *Zea mays*. Cytogenetic evidence has already shown that all four of these genes are in the short arm of the chromosome, that yg_2 is nearest the end and that *wx* is nearest the spindle fibre insertion region.[1,2,3,4,5] In the present study only the position of yg_2 has been determined with any precision, but several points of interest with respect to deletions and deficiencies have arisen. It is with these and the three cases of Yg_2 deficiency in which they appeared that this report deals. One of the deficiencies is small and does not affect the functioning of either the male or the female gametes which possess it. Another shows that a truly terminal deletion can occur. All three give evidence of the position of yg_2.

A stock of maize was used which was homozygous for a large terminal knob on the end of the short arm of a modified chromosome 9 (text figure 1*b*).[1,6] The stock was also homozygous for the dominant allelomorphs of the genes Yg_2, *C*, *Sh* and *Wx*. Pollen from such plants was irradiated and applied to the silks of plants possessing allelomorphs of the above genes (yg_2, *c*, *sh* and *wx*) and a medium-sized knob on the end of the short arm of the normal or "standard" chromosome 9 (text figure 1*a*). The colored, non-shrunken, non-waxy kernels (*C c*, *Sh sh*, *Wx wx*) which were produced by this cross were grown and, as was expected, most of the individuals were normal green (Yg_2 yg_2). A few were yellow-green, indicating a loss or an inactivation of the dominant allelomorph, Yg_2. These plants were examined cytologically.

Examinations were made of the mid-prophase (pachytene) of the first meiotic mitosis in the microsporocytes by the aceto-carmine method. In the normal F_1 plants the synapsis of the short arm of the standard chromosome 9 with that of the modified chromosome 9 brings the medium-sized terminal knob opposite the base of the large one. Since the large knob is approximately five times the size of the smaller one, it protrudes some distance beyond its homologue (text figure 1*c*). The heteromorphism of the knobs, however, in no way affects their close synapsis nor the synapsis of the terminal portion of the short arms.

Deficiency 9-1.—When the microsporocytes of *DF* 9-1 (*DF* for deficiency,

the plant or the chromosome as indicated by the context; 9 for chromosome 9; 1 for the first deficiency reported for that chromosome) were examined it was found that the distal part of the short arm of the modified chromosome 9 was lacking (text figure 1*d*). About one-quarter of its length was lost as determined by measurements of camera lucida drawings of several figures. Synapsis of the short arms from the spindle fibre insertion regions to the point of deletion was regular in most cases. Sometimes there was a lack of association of the last few chromomeres of the deficient chromosome with their homologues on the standard chromosome. Beyond the point of deletion the standard chromosome was frequently more or less folded back on itself. In order to make it as nearly certain as



TEXT FIGURE 1

a. Diagram of chromosome 9 of *Zea mays*, possessing a medium-sized knob on the end of the short arm. Spindle fibre insertion region indicated by cross bar.

b. Diagram of modified chromosome 9 with large knob on end of its short arm.

c. Diagram of normal F_1 from cross of the two chromosomes shown in (*a*) and (*b*).

d. Diagram of an exceptional F_1, Deficiency 9-1, showing loss of the terminal portion of modified chromosome 9.

e. Diagram of Deficiency 9-2 showing loss of an internal segment near the end of the short arm of modified chromosome 9.

f. Diagram of Deficiency 9-3 showing deletion of part of the large knob and a small part of the short arm of modified chromosome 9.

possible that the part deleted was a terminal and not an internal segment of the arm careful study was made of the end of the deficient chromosome for some evidence of knob material. If an internal deletion had occurred, it would be expected that the knob would be visible at the end of the deficient chromosome (see *DF* 9-2). Since this knob can be partitioned in x-ray treatment (*DF* 9-3) it might be represented by only a small chromomere-like part. As no knob was discerned, it is highly probable that the deletion was truly terminal. It might be pointed out that in previously reported deletions (*Drosophila*, *Zea*, etc.) the lack of any distinctive morphological appearance of the end of the chromosome has made it impossible to decide whether a terminal segment or an internal segment near the end had been deleted. In fact, there has been no critical evidence for the occur-

rence of terminal deletions. Studies of chromosome morphology at diakinesis or metaphase or the detection of the loss of the terminal genes of a linkage group cannot substitute for a knowledge of the changes in the minute morphology of the prophase chromosomes in such a question.

The plant designated as *DF* 9-1, although partly yellow-green, had some normally colored regions. This means that the deletion did not occur in the male nucleus nor the first embryonic division but happened at some later stage in the development of the embryo. Only the tissue which was derived from one of the daughters of the cell in which the deletion occurred showed the recessive character. The proportion of the plant body which was produced by this sector diminished as the plant grew. Half of the first and second leaves, a stripe on the third, fourth and fifth leaves, and no part of the succeeding leaves were yellow-green. Only one branch of the tassel gave any evidence of deficiency. Since all of it was used for cytological examination, no pollen was available for genetic study of the extent of the deficiency (that is, whether *C*, *Sh* or *Wx* were lost with *Yg*). No kernels developed to maturity, their failure probably being due to seasonal conditions.

Deficiency 9-2.—The plant designated as *DF* 9-2 was found to have an internal deletion in the short arm of the modified chromosome 9. About a third of the arm was lost but the large knob was still present (text figure 1*e*). Near the spindle fibre insertion region synapsis was regular, but in the more distal regions near the large knob asynapsis was frequent. Sometimes the medium-sized knob was associated with the large one, necessitating a bulging or folding back of the standard chromosome to accommodate for the difference in the lengths of the two chromosomes. From the configurations observed it is clear that the part deleted was near the end of the short arm close to the terminal knob.

It has been pointed out previously that a distinction between the deletion of the terminal region of a chromosome and a deletion of an internal segment near the end of a chromosome may not be possible from studies of synaptic behavior alone, but may require the presence of distinctive terminal morphological features.[7] In the cases described above the difference in the synaptic configurations might have been sufficient to make a decision possible, but in each case a relatively large segment was deleted. The presence of the knobs, however, facilitated the solution and made it more decisive. It seems clear that in many cases of small deletions (see *DF* 9-3) only terminal visible markers will render the distinction possible.

The plant called *DF* 9-2 was completely yellow-green and was nearly as vigorous as a homozygous recessive yellow-green plant. It shed no pollen, however, nor were any silks exserted. Again it was impossible to determine whether any of the marked genes were lost with Yg_2.

Deficiency 9-3.—The third case of Yg_2 deficiency was a plant fully as vigorous as a homozygous recessive yellow-green individual. Examination of the microsporocytes showed that a very small internal deletion had occurred (text figure 1*e*). The evidence for this is that the terminal knob on the modified chromosome 9 was slightly smaller than the normal large knob. The synapsis of the short arm of chromosome 9 with the modified 9 was complete and regular. The association of the ends of the chromosomes in diplotene was normal, showing none of the precocious opening out which is often evidence of lack of complete homology. At diakinesis the configurations were entirely normal. Thus, from observations alone it would seem that the deletion included only a part of the knob. However, it seems more probable that a small portion of the arm of the chromosome was deleted also. If only a part of the knob were lost, then the locus of yg_2 must be in the knob, unless simultaneous deletion and gene mutation or inactivation are assumed. That the locus is not in the knob is evident from the fact that crossing-over can occur between the knob[4] and the gene and from the fact that in *DF* 9-2 there was no apparent diminution of the size of the large knob concurrent with the loss of Yg_2. Hence, although it is possible that Yg_2 mutated to the recessive form coincident with the deletion it is also reasonable to infer that the deletion included some of the adjacent chromatin carrying the locus of Yg_2 with part of the knob. This, in agreement with other data, places the locus of the gene yg_2 very near to the end of the short arm of chromosome 9.

The majority of genetical and cytological studies of plants have shown that haploid gametes or gametophytes lacking any part of their chromosome set are not functional. Recently there have been found some exceptions in which deficient gametes were viable.[8,9]

The first indication that the case here described was an exception came when it was found that there were equal proportions of *Wx* and *wx* pollen grains (in one count, 385 *Wx*: 381 *wx*). In this plant the deficient chromosome carried *Wx*. It had been thought that only a few, if any, *Wx* grains would be found, these being the ones possessing a chromosome which resulted from a crossover between the locus of *Wx* and the deletion. Such grains would have no deficiency. The *Wx* grains were of the same size as the others and equally well filled with food material. That they were equally viable and functional in competition with the non-deficient grains was shown by the fact that backcrosses to the multiple recessive gave no deficiency in the number of colored, non-shrunken, non-waxy kernels. There were 9 *C Sh Wx*, 3 *C Sh wx*, 1 *C sh wx*, 1 *c Sh Wx*, 6 *c sh Wx* and 16 *c sh wx* kernels. The reciprocal cross of the deficient plant by the multiple recessive showed that the female gametophytes and gametes possessing the deficient chromosome function to produce normal-sized, well-developed kernels. Here there were 47 *C Sh Wx*, 12 *C Sh wx*, 3 *C sh wx*, 5 *c Sh Wx*,

19 *c sh Wx* and 34 *c sh wx* kernels. Crossing-over between the loci of *C*, *Sh* and *Wx* occurred with a frequency equal to that found in normal material. When some of the colored, non-shrunken, non-waxy (*C c, Sh sh, Wx wx*) and colorless, shrunken, waxy (*c c, sh sh, wx wx*) kernels were grown, it was found that crossing-over had occurred in the region between the locus of *C* and the deficiency. All of the plants were yellow-green whether they possessed the gene yg_2 in the simplex (heterozygous for deficiency) or in the duplex condition (homozygous for the recessive). Counts of the pollen from the middle part of single anthers of plants heterozygous for the deficiency showed that there was no more abortion of pollen grains than is normal for maize. In one case 92 out of 1694 (5.4 per cent) were undeveloped. Pollen from one of the plants heterozygous for the deficiency and for the genes *C, Sh* and *Wx* was used in crosses to the multiple recessive stock the results agreeing with those obtained in the previous backcross (171 *C* : 165 *c*). It was possible to self-pollinate only one individual heterozygous for the deficiency. Unfortunately, it was homozygous for recessive *c, sh* and *wx*. This makes necessary a cytological examination of the plants from the kernels obtained for the presence of the diminished knob in the homozygous condition before it can be known whether a homozygous deficiency is viable.

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[3] Creighton, H. B., and McClintock, B., *Ibid.*, **17**, 491–497 (1931).
[4] Creighton, H. B., unpublished.
[5] Burnham, C. R., unpublished.
[6] McClintock, B., *Proc. Nat. Acad. Sci.*, **16**, 791–796 (1930).
[7] Idem, *Zeitschr. Zellforsch. Mikr. Anat.*, **19**, 192–237 (1933).
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NOTES ON INTENSITIES IN THE SPECTRUM O II

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Note on Intensities in the Spectrum O II.—When a discharge is studied in detail, anomalies are often encountered in the intensities of the spectra emitted which require for their elucidation rather full information about the electrical state of the emitting tube, but can in general be then referred to known atomic or molecular properties. In some cases, however, when the discharge parameters are adequately understood, known atomic properties are found insufficient to account for what is observed, and an

indication is obtained of the existence of some new atomic effect, the further study of which requires that it be isolated as far as possible from the multiplicity of processes usually coexistent with it. The following note deals with some anomalous intensities in the spectrum O II, which lead to speculations about the chance of simultaneous dissociation, ionization and excitation of the molecule O_2 by electron impact.

This spectrum is readily obtained from the negative glow of a cold cathode glow discharge,[1] and from the positive column of a condensed discharge.[2,3] The intensity distribution in the latter case is similar to that observed in absorption in type *B* stars.[4] With a negative glow of low current density, some multiplets are missing.[1] Two lie in the visible, at 4956 – 4941 ($(^3P)\ 3p\ ^2P^0 - (^3P)\ 3d\ ^2D$) and 4406 – 4359 ($(^3P)\ 3p\ ^2D^0 - (^3P)\ 3d\ ^2D$), and one in the ultra-violet at 2419 – 2407 ($(^3P)\ 3p\ ^2D^0 - (^1D)\ 3d\ ^2D$). The 4950 multiplet is now added to those previously given,[1] from a study of plates more sensitive to this region. The multiplets do not appear in the glow even when other lines of O II which are of comparable intensities in the positive column spectrum are heavily overexposed.

In an attempt to arrive at some explanation of this anomaly, spectra have been examined from three other sources:

(1) A hollow cathode discharge at 440 volts, 2 milliamps per cm.^{2}, and 1 mm. pressure.

(2) An electron beam discharge in an equipotential box, at 110 volts, 0.02 mm. pressure.

(3) Some arc and glow discharges from, mainly, a hot cathode.

For the first two, Pyrex tubes were built in which all wax joints were avoided in order to remove any suspicion that earlier results might have been connected with presence of a trace of carbon. For the last, the plates obtained by Lockrow[5] were reëxamined. In all three cases only the visible multiplets were studied. They were obtained faintly with long exposure in (1), and were again missing in (2) and (3).

The following causes for absence of the lines are excluded: (i) Inadequacy of electron energies. This has already been dealt with,[1] and that the electron energies are adequate is also shown nicely by the occurrence of the 4185–4189 multiplet in considerable strength on all plates. This multiplet requires almost the same energy for excitation of its initial term, $(^1D)3d\ ^2G$, as the initial term of the missing u.v. multiplet.

(ii). The lines have a sharply peaked excitation function. Lockrow's discharges covered a potential range from 50–350 volts, and so the peak would have to lie at a higher voltage, which is extremely unlikely. Moreover, some trace of the lines would be expected on long exposure if there were a lower voltage peak.

This leaves the possibility that, in the heavier current discharges (hollow cathode, positive column), they are produced as a result of some

cumulative effect, probably successive electron impacts. Somewhat similar but less pronounced differences in the spectrum Ar II have been attributed by Druyvesteyn[6] to excitation by electron impact from the neutral atom at low current, and to excitation from the singly charged ion at high current. We can only speculate as to the stages in the present case when the full spectrum appears, but it is possible, from the appearance of the full spectrum in type *B* stars, that the first stage is a dissociation, and the second ionization and excitation from a neutral oxygen atom in the ground state or one of the two low metastable states. Cumulative processes would be favored by high current, and the absence of certain multiplets at low current may be attributed to failure to excite their initial states in a single act from the neutral molecule as the starting point. At the same time it is difficult to see why a ^2G and a ^2D term of almost identical value, based on the same state of the core O^{++}, should have such different excitation properties, and, with a less well analysed spectrum, one might be tempted to suppose that the term analysis was at fault.

Irrespective of the explanation, the present work serves to emphasize that in seeking to find a standard distribution of intensities in a spectrum such as O II,[4] the state and degree of dissociation of the molecules has to be taken into account, as well as the factors important in the case of a monatomic gas.

Miss D. Scott obtained the spectra with sources (1) and (2), and Lockrow's plates were kindly lent by Professor O. S. Duffendack, whose interest in this work is gratefully acknowledged.

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[6] Druyvesteyn, *Zeits. Physik*, **64**, 781 (1930).

ON THE VECTOR MODEL FOR ALMOST CLOSED SHELLS

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In discussing atomic spectra it is often convenient to introduce the idea of a hole.[1] If a shell is more than half filled with equivalent electrons and if the properties of this group of electrons can be formulated in terms of the corresponding group of holes, there is clearly an advantage in dealing with the latter rather than with the more numerous electrons. It is therefore

of interest to see how the properties of a group of equivalent electrons may be formulated in terms of the corresponding holes. We will be particularly interested in determining, from this point of view, to what extent vector model considerations are applicable to holes.

The formulation of quantum mechanics in terms of quantized amplitudes is especially convenient for the introduction of holes,[2] for the variables then refer directly to the number of electrons in the different states. It has been shown[3] that if we are dealing exclusively with the states arising from one electron configuration (which is determined by assigning definite values to M_i, the number of electrons in the ith shell), then an operator in coördinate space of the type $F^0 = \sum_k f^0(x_k)$ may be written in terms of quantized amplitudes as

$$F = \sum_k f_k$$

$$f_k = \sum_{pr}(k)\ a_p^+ a_r (p|f^0|r).$$

Here $\sum(k)$ means a summation limited to the states of the kth shell. In this way it was demonstrated that the total orbital angular momentum $\mathbf{L}$ and the total spin angular momentum $\mathbf{S}$ could be decomposed into $\sum_k \mathbf{l}_k$ and $\sum_k \mathbf{s}_k$, where $\mathbf{l}_k$ and $\mathbf{s}_k$ satisfied such commutation relations that they could be taken to represent the resultant orbital and spin momentum of the electrons in the kth shell.

Suppose we wish to deal with the holes of the jth shell. The transformation equations are[4]

$$\bar{a}_r^+ = a_r \quad \bar{a}_r = a_r^+ \text{ if } r \text{ is in the shell } j$$

$$\bar{a}_r^+ = a_r^+ \quad \bar{a}_r = a_r \text{ if } r \text{ is not in the shell } j.$$

The $\bar{a}$'s satisfy the same commutation relations as the a's and $\bar{N}_r$ is zero for the states of j that are filled, and one for those that are empty. Define

$$\bar{f}_k = \sum_{pr}(k)\ \bar{a}_p^+ \bar{a}_r\ (p|f^0|r).$$

Using the commutation relations for the $\bar{a}$'s and the above transformation equations we find

$$f_k \longrightarrow \bar{f}_k \qquad k \neq j$$

$$f_j \longrightarrow \sum_p (j)\ (p|f^0|p) - \bar{f}_j^*.$$

If f is an angular momentum, it is easily verified that $\sum_p (j)\ (p|f^0|p) = 0$. Hence

$$\left.\begin{matrix} \mathbf{l}_k \longrightarrow \bar{\mathbf{l}}_k \\ \mathbf{s}_k \longrightarrow \bar{\mathbf{s}}_k \end{matrix}\right\} k \neq j \qquad \begin{matrix} \mathbf{l}_j \longrightarrow -\bar{\mathbf{l}}_j^* \\ \mathbf{s}_j \longrightarrow -\bar{\mathbf{s}}_j \end{matrix}.$$

Thus we must use minus the complex conjugate for the resultant angular momenta of the holes in the jth shell if we wish to represent the resultant angular momenta of the corresponding group of electrons. Since $-\bar{\mathbf{l}}_j^*$ and $-\bar{\mathbf{s}}_j^*$ obey the same commutation relations with one another and with the other angular momenta as $\bar{\mathbf{l}}_j$ and $\bar{\mathbf{s}}_j$, this change does not affect the compounding of these vectors.

Let us next briefly consider the question of intensities for transitions in which an electron jumps from shell i to shell j. The electric moment is[5]

$$\mathbf{p} = \sum_{pr} (i \text{ and } j) a_p^+ a_r \, (p|\mathbf{p}^0|r) = \sum_p (i) \sum_r (j) \{a_p^+ a_r (p|\mathbf{p}^0|r) + a_r^+ a_p (r|\mathbf{p}^0|p)\}.$$

On transforming to the holes

$$\mathbf{p} \longrightarrow \sum_p (i) \sum_r (j) \{\bar{a}_p \bar{a}_r^+ \, (p|\mathbf{p}^0|r) + \bar{a}_r \bar{a}_p (r|\mathbf{p}^0|p)\}.$$

Now this satisfies the same commutation relations with $\bar{\mathbf{l}}_i - \bar{\mathbf{l}}_j^*$ and $\bar{\mathbf{s}}_i - \bar{\mathbf{s}}_j^*$ as $\mathbf{p}$ does with $\mathbf{l}_i + \mathbf{l}_j$ and $\mathbf{s}_i + \mathbf{s}_j$. Consequently the transition probabilities are governed by the usual formulas in so far as they depend on the sum of the angular momenta for the two shells i and j, and on the angular momenta of the other shells. For example, in transitions from $p^5 d$ to p^6 the intensities in LS coupling are given by the Kronig factors in so far as L, S and J are concerned.

Finally we turn our attention to the calculation of resonance energies. We will make a comparison of the effect of resonance on the interaction between two non-equivalent electrons with its effect on the interaction between a hole and an electron. Let the subgroups involved be i and j. Then from reference 4, remembering $M_i = M_j = 1$, the electrostatic energy is

$$H = \sum_{pr} (i) \sum_{qs} (j) a_p^+ a_q^+ a_s a_r [(pq|1/r_{12}^0|rs) - (pq|1/r_{12}^0|sr)]_2$$

where the second term representing exchange is the one in which we are interested. Let O^0_{12} be an operator in coördinate space which, when applied to a function of the electronic coördinates 1 and 2, interchanges these coordinates. Dirac has shown[6] that the dependence of O_{12}^0 on the spin coordinates is $1/2(1 + 4\mathbf{s}_1^0 \cdot \mathbf{s}_2^0)$ so

$$O_{12}^0 = Q_{12}^0 1/2(1 + 4\, \mathbf{s}_1^0 \cdot \mathbf{s}_2^0)$$

where Q_{12}^0 depends only on the spacial coördinates. The resonance term in which we are interested is

$$H_r = - \sum_{pr} (i) \sum_{qs} (j) \; a_p^+ a_q^+ a_s a_r (pq|1/r_{12}^0 \quad Q_{12}^0 1/2(1 + 4\mathbf{s}_1^0 \cdot \mathbf{s}_2^0)|rs).$$

Now it may be proved in a manner very similar to the deduction of Eq. (24) of reference 3 that with the restriction $M_i = M_j = 1$,

$$H_r = -\{\sum_{pr}(i)\sum_{qs}(j)a_p^+a_q^+a_sa_r\ (pq|1/r_{12}^0\quad Q_{12}^0|rs)\}1/2(1+4\ \mathbf{s}_i\cdot\mathbf{s}_j)$$
$$= -H_r^1 1/2(1+4\ \mathbf{s}_i\cdot\mathbf{s}_j).$$

Here H_r^1 is independent of the electron spin. In LS coupling

$$1/2(1+4\ \mathbf{s}_i\cdot\mathbf{s}_j) = \begin{cases} 1 & \text{for triplets} \\ -1 & \text{for singlets} \end{cases}$$

We see therefore that for two electrons in LS coupling the resonance energy $-H_r^1$ (which still depends on L) is added for the triplets and subtracted for the singlets.

Suppose now that there is only one electron missing from the ith shell. On transforming to the holes

$$H_r \longrightarrow \sum_{pr}(i)\sum_{qs}(j)\bar{a}_p^+\bar{a}_q^+\bar{a}_s\bar{a}_r\ (rq|1/r_{12}^0\quad Q_{12}^0 1/2(1+4\ \mathbf{s}_1^0\cdot\mathbf{s}_2^0)|ps).$$

Again using the restriction $\overline{M}_i = \overline{M}_j = 1$ it can be shown

$$H_r \longrightarrow \overline{H}_r^1\ \sum_{pr}(i)\sum_{qs}(j)\bar{a}_p^+\bar{a}_q^+\bar{a}_s\bar{a}_r\ (rq|1/2(1+\mathbf{s}_1^0\cdot\mathbf{s}_2^0)|ps)$$
$$\longrightarrow \overline{H}_r^1 1/2\ (1+4\bar{\mathbf{s}}_i^*\bar{\mathbf{s}}_j).$$

Now in coupling together the hole and the electron the resultant spin is $-\bar{\mathbf{s}}_i^* + \bar{\mathbf{s}}_j$ so that $-2\bar{\mathbf{s}}_i^*\cdot\bar{\mathbf{s}}_f = S^2 - 3/2$. Hence in LS coupling

$$1/2(1+4\ \bar{\mathbf{s}}_i^*\cdot\bar{\mathbf{s}}_j) = \begin{cases} 0 & \text{for triplets} \\ 2 & \text{for singlets} \end{cases}$$

Thus the resonance for all the triplets is zero, a result which is confirmed in the cases that have been computed.[4]

In conclusion it is the writer's pleasant duty to acknowledge his indebtedness to discussions with Professor Breit and Dr. Hall.

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[1] By a hole we mean an electron missing from a partially filled shell. Thus, corresponding to a group of eight equivalent d electrons, there are two equivalent d holes.

[2] W. Heisenberg, *Ann. Physik*, **10**, 888 (1931).

[3] M. H. Johnson, Jr., *Phys. Rev.*, **43**, 627 (1933).

[4] M. H. Johnson, Jr., *Ibid.*, **43**, 632, Eq. 5 (1933).

[5] M. H. Johnson, Jr., these PROCEEDINGS, **19**, 916, Eq. 8 (1933).

[6] P. A. M. Dirac, *Principles of Quantum Mechanics*, Oxford Press (1930), § 66.

CAN MAMMALIAN EGGS UNDERGO NORMAL DEVELOPMENT IN VITRO?*

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The certain test of the viability and normal development of mammalian ova subject to experimental manipulation in vitro may be made by transplanting these ova into a properly prepared female and obtaining young which have developed from the transplanted ova. Heape,[1] Biedl[2] and Pincus[3] long ago demonstrated that rabbit eggs in the early cleavage stages may be successfully transplanted from one female to another (cf. Nicholas[4]). Pincus[3] had no success in obtaining young from culture-grown rabbit ova, and the conclusion was that either greater refinement of technique was required or that apparently "normal" development in vitro was not attained. Thus the regular sequence of cleavage divisions observed in fertilized ova placed in appropriate culture media may be abnormal in the sense that ova which so cleave might never give rise to rabbits. Similarly the observation of apparently successful fertilization of mammalian ova in vitro (Long,[5] Pincus,[3] Yamane[6]) is also open to question. For it has been demonstrated (Yamane,[6] Gilchrist and Pincus[7]) that mammalian eggs placed with dead spermatozoa will exhibit the usual signs of fertilization, e.g., shrinkage of the vitellus and formation of the second polar body. In addition, the parthenogenetic cleavage of rabbit ova occurs readily under certain culture conditions (Pincus[3]).

We have in two instances obtained young from ova subjected to experimental manipulation in vitro.

In one case ten ova were recovered from the fallopian tubes of an agouti doe mated to a vasectomized English-spotted buck. These ova were obtained 13 hours after copulation and were not fertilized since vasectomy had rendered the male sterile. Evidence of the fact that no fertilization had taken place was had by the presence of typical follicle cell masses about each ovum (see Yamane,[6] Pincus,[3] Pincus and Enzmann[8]). Sperm obtained from the vas deferens of a self-colored non-agouti black male were placed with those ova for a period of twenty minutes. During this period the events ordinarily occurring during insemination were observed, namely, the falling away of the surrounding granulosa mass and the slight shrinkage of the vitellus. The eggs freed of surrounding spermatozoa were taken up by pipette in a small amount of Ringer's solution and transferred to the right fallopian tube of a New Zealand Red doe rendered pseudopregnant by mating 48 hours previously with a vasectomized English-spotted male.. Thirty-three days later the New Zealand doe

produced seven dark gray young. If, by some chance, the vasectomized male had produced spermatozoa these young would have been English spotted. Since the ova of pseudopregnant does are certainly not fertilizable by twenty-four hours after copulation (Hammond,[9] Pincus[3]), and since the New Zealand Red foster mother was mated not twenty-four but forty-eight hours previous to the transfer of ova, it is obvious that the young could not have arisen from the adventitious fertilization of her ova by a few spermatozoa that may have been accidentally introduced into the tubes with the transplanted ova. We believe, therefore, that this is the first certain demonstration that mammalian eggs can be fertilized in vitro.

In a second case a pseudopregnant albino doe (rendered pseudopregnant by copulation with a vasectomized black male) received five ova which had been cultured in Carrel flasks for twenty hours before their transplantation to the left fallopian tube of the pseudopregnant doe. These ova had been obtained in the one cell stage from the fallopian tubes of an English doe mated to a fertile agouti male. Two gray English-spotted young were obtained thirty-one days after the transplantation.

We believe that the successful recovery of young in these cases (in contrast to the negative results previously reported) is due to careful refinements of the operative procedure which will be described in detail elsewhere.

Incidentally, these experiments demonstrate that the corpora lutea of pseudopregnancy are fully functional.

* The experiments here reported have been supported in part by a grant from the NATIONAL RESEARCH COUNCIL Committee for Research in Problems of Sex.

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ACHROMOBACTER ICHTHYODERMIS, N. SP., THE ETIOLOGICAL AGENT OF AN INFECTIOUS DERMATITIS OF CERTAIN MARINE FISHES

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Communicated January 15, 1934

For several months the experimental work in fish biology (see Wells, 1932) at the Scripps Institution of Oceanography has been hampered by a highly fatal dermal infection of certain marine fishes. Approximately nine-tenths of the Pacific killifish, *Fundulus parvipinnis*, brought into the laboratory aquaria have died of the disease. Gobies, blennies, smelt and other fishes maintained in aquaria have exhibited similar lesions. These and other species of marine fishes which we have successfully infected with material from naturally occurring lesions, as well as details of the pathology and epidemiology of the disease will be discussed elsewhere. That the disease occurs in nature is certain because several infected *Fundulus* have been observed in their native habitat.

In the case of *Fundulus* the first appearance of the disease is a whitish spot on the dorsal or lateral part of the body between the operculum and caudal fin. This rapidly increases in extent and may involve the entire dorsolateral surface of the fish. One of the first histological manifestations is the disintegration of melanophores and the destruction of pigmentary bodies, which probably accounts for the peripheral blanching. The scales are raised (Fig. 1) and gradually slough off. The central area of the lesion becomes flecked with red. Unless specially treated the mortality has been near 100 per cent.

Microscopically, direct smears from the *Fundulus* lesions reveal the presence of an almost pure culture of the bacterium described below (Fig. 2). After the lesion has progressed three to six days numerous ciliates usually appear on the scene and there is a diminution in the number of bacteria in the superficial layers of the lesion. At times practically no bacteria can be demonstrated in the diseased tissue, although invariably cross inoculations prove their presence. In the later stages of the infection the ciliates disappear and externally the lesion becomes butyrous with bacteria. Tissue sections show that the bacteria first invade the dermal layers and gradually attack the muscle tissue. Apparently necrosis precedes their inward advance. Bacteria are not found intracellularly but occur in sheets in the interstitial spaces of the diseased tissue. Frequently in the advanced stages of the infection secondary invasion of gram-negative bacilli occurs. There is no evidence of the internal organs being involved nor has it been possible to demonstrate the causative bacteria in the blood.

The etiological agent has been repeatedly isolated in pure culture from the lesions of naturally infected *Fundulus*. Healthy fishes are readily infected with it by merely placing a loopful of the culture under a scale. Controls similarly treated mechanically or even with an attenuated strain remain normal. The incubation period is very short since fishes develop lesions in 4 to 18 hours after inoculation, and may die in a week or two. Several times the same bacterium has been reisolated in pure culture from artifically infected fishes, this used to inoculate other fishes and the process repeated. The virulence of the invading microbe is maintained by such passage. The mortality of fishes infected with pure culture is not as high as that of fishes infected directly from lesions, although many of the former succumb.

The bacterium involved is a gram-negative bacillus measuring 0.9 to

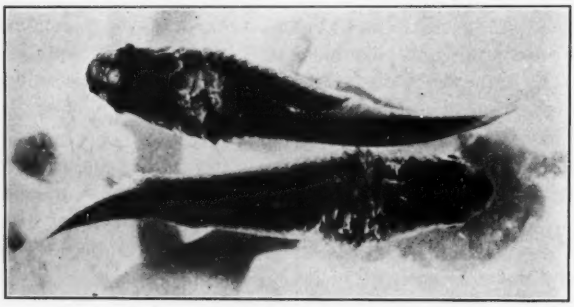

FIGURE 1

Lesions on *Fundulus parvipinnis* five days after they had been inoculated with a loop of material from a naturally-occurring *Fundulus* lesion. Fishes 8 cm. in length.

1.3 μ in width, by 3 to 5 μ in length. No spores are formed. It is encapsulated both in tissue and on artificial media. It is actively motile by means of a tuft of polar flagella. The bacilli occur singly or in pairs and only infrequently in short chains. The morphology of the bacilli from the lesions or from recently inoculated cultures is very constant, but pleomorphism is the rule in cultures which are a few hours old. In such cultures bent rods and other bizarre forms predominate. Virtually no recognizable forms are found in cultures a few days old, although upon inoculation of susceptible fish with this material, bacilli of the originally described morphology appear as infection progresses.

The organism grows rapidly on sea-water peptone agar, forming glistening, convex, circular colonies which are 2 to 4 mm. in diameter. On slants of such media there is an abundant, filiform, raised, smooth opalescent growth. Crateriform liquefaction of gelatin begins the first day

and is complete in 5 days at 18°C. In sea-water nutrient broth the organism multiplies with unusual rapidity, making the media turbid in 3 to 4 hours at 25°C. Under these conditions a pellicle is formed, granular sediment is scanty and odor is absent. The organism does not grow in milk nor will it multiply in any of the conventional fresh-water media, although it thrives in similar media prepared with sea water. In fact, when suspended in fresh-water media, it dies after a few hours. In these respects it is typical of marine bacteria the majority of which, as pointed out by ZoBell and Feltham (1933), require sea-water substrata and fail

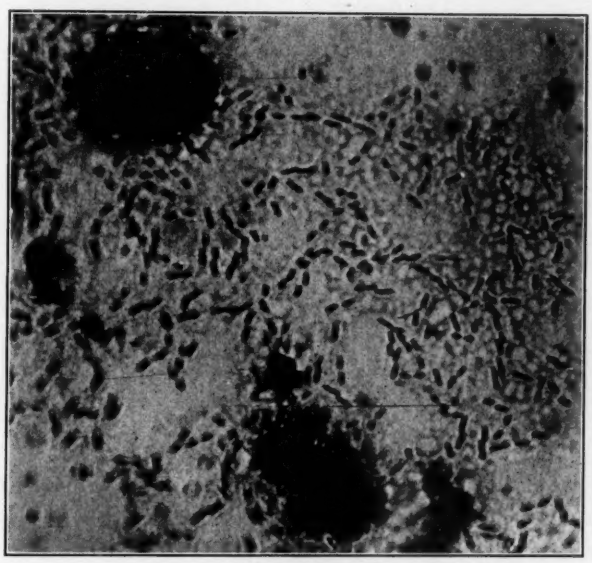

FIGURE 2

Impression smear from a lesion of a naturally infected *Fundulus parvipinnis*. (× ca 2000.)

to reproduce in fresh-water media. There is no growth on ordinary potato slants, but on potato dialyzed in sea water the bacilli multiply. Pigment formation has not been observed, although the cultures have a tendency to darken slightly with age.

Standard differential media (Committee, 1933) were used in characterizing the bacterium with the exception that sea water was substituted for fresh water. It hydrolyzes starch and ferments glucose, sucrose and mannitol with the formation of acid to pH 6.0, but no gas is evolved. Lactose and glycerol are not attacked. In peptone media the reaction becomes more alkaline as ammonia is produced. Hydrogen sulphide pro-

duction is negative. Indol is formed in tryptophane broth. Nitrates are reduced to nitrites, and nitrites are not attacked. The bacterium is primarily an aerobe but develops anaerobically in semisolid media containing glucose, maltose or starch. The optimum hydrogen-ion concentration is about pH 7.8 – 8.2.

The organisms survive ten minutes at 40°C. but are killed at 45°. Viability is destroyed by a few hours' incubation at 37° and, in some experiments, by overnight incubation at 33.5°. They reproduce at 30° and at temperatures as low as 4°C., their optimum for multiplication being from 25° to 30°C. The maximum virulence for *Fundulus* is exhibited when the cultures are incubated at 20° to 25°C. Cultures grown at temperatures much in excess of 25° or below 15° rapidly lose their infectivity. However, fish held at temperatures from 5° to 30° may develop the disease when inoculated, but the maximum progression of lesions occurs when the fish are maintained in water at 15° to 25°C. Though the onset of the disease is slower in fishes at temperatures below 15°, the mortality is not diminished. On the other hand, fishes kept in water above 30° are not infected when inoculated, and badly diseased fish, if successfully acclimated to temperatures of 32° and 35°C., rapidly and completely recover. On artificial media the bacilli can adapt themselves to multiply at higher temperatures. By raising the incubation temperature increments of one-half degree and inoculating the cultures while in the logarithmic increase growth phase, the bacilli have been acclimated to multiply at 38.5°. This seems to be the critical point because, even though the temperature was increased by as little as two-tenths-degree steps, 38° to 38.5° could not be exceeded.

Our examination of the available literature, including the standard works on fish diseases by Hofer (1906) and Plehn (1924), has failed to reveal any pathological process which resembles the description of this *Fundulus* dermatitis. Nor is the bacterium described by Bergey (1930). The generic characteristics of the bacterium indicate that it is an *Achromobacter*, and since it is obviously a new species, the name *ichthyodermis* is proposed.

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MINIMUM NUMBER OF SQUARES IN A GROUP WHEN NOT ALL OF THEM ARE RELATIVELY COMMUTATIVE

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It has recently been proved[1] that whenever a group G involves less than nine operators which are squares under it then all these squares are always relatively commutative, and it is obvious that the tetrahedral group involves nine operators which are squares under it and which are not all relatively commutative. In the present article it is proved incidentally that if a group contains exactly nine operators which are squares under it but are not all relatively commutative then it is either the tetrahedral group or the direct product of this group and the abelian group of order 2^m and of type $(1,1,1,\ldots)$. The main object of this article is to establish some theorems relating to the smallest number of squares which can appear in G when it involves at least two squares which are not relatively commutative, and to establish two theorems relating to the possible subgroups composed of the squares of a group .

Suppose that s_1 and s_2 represent two operators of order 2 which are squares under G but are not relatively commutative. The smallest possible order of the dihedral subgroup generated by s_1 and s_2 is 6. We shall first consider the case when the order of this subgroup is 6 and shall prove that G must then involve at least 18 operators which are squares under it. Moreover, there is a group of order 36 in which these conditions are satisfied and which can be represented as a transitive permutation group of degree 6. It is formed by the positive permutations of the transitive group of degree 6 and of order 72.

Let t_1 represent an operator of G whose square is s_1. Its powers transform the operator of order 3 which is equal to s_1s_2 into four distinct operators since t_1^2 transforms it into two such operators. Hence G involves at least four subgroups of order 3. If it involves more than four such subgroups there are clearly more than 18 operators in G which are squares thereunder. If it involves exactly four such subgroups and these generate a group of order 9 then s_1 transforms all of its operators into their inverses and hence G involves at least 18 operators which are squares thereunder since the conjugates of s_1 have this property. If these four subgroups would not generate a group whose order is divisible by 9 they would be transformed under G according to the alternating or the symmetric group of degree 4 and hence G could not involve an operator whose square would transform s_1s_2 into its inverse. This proves the following theorem: *If a group involves two non-commutative operators of order* 2 *which are squares*

thereunder and whose product is of order 3 *then it involves at least* 18 *operators which are squares thereunder.*

When s_1 and s_2 generate the octic group then the squares of the operators of G generate a subgroup whose order cannot be less than 16 since this octic group is not transformed into itself by the operator of G whose square is s_1. The order of this invariant subgroup cannot exceed 16 when G is the smallest group which involves operators satisfying the conditions imposed on s_1 and s_2 since we can construct a group of order 64, which contains such an invariant subgroup of order 16 and satisfies the conditions imposed on G. This construction can be effected as follows: Form a group of order 32 by establishing a (4,4) correspondence between two octic groups with respect to their cyclic subgroups and extend the group thus obtained by an operator of order 4 which is the product of the operator of order 2 which merely interchanges corresponding operators of these octic groups and a non-invariant operator of order 2 in one of these octic groups.

From the preceding paragraph it results that it may be assumed that G contains the group of order 16 obtained by establishing a (2,2) isomorphism between two octic groups as the invariant subgroup generated by its operators which are squares under it. By extending this subgroup by an operator whose square is s_1 there results a group of order 32 which does not contain an operator whose square is s_2. Hence this group must again be extended to obtain G. As similar remarks apply to the groups which involve two non-commutative operators of orders 2 and 4, respectively, which are squares thereunder as well as those which involve two such operators of order 4 there results the following theorem: *If a group contains two operators which are squares thereunder and which generate either the octic or the quaternion group then it involves at least twelve operators which are squares under it and its order is at least* 64. *There are three distinct groups of order* 64 *which correspond, respectively, to the three cases when the two given operators are of order* 2, *of order* 4 *or when one is of order* 2 *and the other of order* 4.

If a group contains two operators of order 2 which are squares thereunder and generate the dihedral group of order 10 it contains at least ten operators which are squares under it and the holomorph of the group of order 5 involves exactly ten such squares. When s_1 and s_2 generate a dihedral group whose order exceeds 10 G obviously contains more than twelve operators which are squares under it. When G involves two non-commutative operators of order 3 they are necessarily squares under it and they may generate the tetrahedral group which involves nine operators which are squares under it. They may also generate the group of order 24 which involves no subgroup of index 2 and contains ten operators which are squares under it. In all other cases they generate a group which involves at least twelve operators which are squares under it. If an operator of order 3 and an operator of a larger order which is not commutative with

this operator of order 3 are squares under a group it clearly contains more than twelve operators which are squares.

It was noted above that if G contains two operators of order 2 and 4, respectively, which are squares under it and generate the octic group then it involves at least twelve operators which are squares under it, and it is easy to see that when it involves two non-commutative operators which are squares under it such that one is of order 2 while the order of the other exceeds 4 then it must also involve more than twelve operators which are squares under it. As this is also the case when it involves two non-commutative operators of order 4 which are squares under it but do not generate the quaternion group it results that when a group involves no more than nine operators which are squares under it all of these squares are relatively commutative except when this group is either the tetrahedral group or the direct product of this group and the abelian group of order 2^m and of type (1,1,1,...).

Every abelian group is composed of operators which are squares under an abelian group as well as under the non-abelian group obtained by extending the latter by an operator of order 2 which transforms each of its operators into its inverse. On the other hand, it was noted above that neither the octic group nor the quaternion group can be composed of all the operators which are squares under a group. We proceed to prove the following theorem: *A necessary and sufficient condition that a dihedral group is composed of the operators which are squares under a group is that the order of its cyclic subgroup of index* 2 *is not divisible by* 4 *and that there are numbers which belong to exponent* 4 *with respect to each odd prime factor of this order.* A proof of this theorem results directly from the facts that a necessary and sufficient condition that the group of isomorphisms of the cyclic group of order p^m, p being an odd prime number, involves an operator of order 4 whose square transforms every operator of this cyclic group into its inverse is that $p-1$ is divisible by 4, and that the group of isomorphisms of the cyclic group of order 2^m involves no operator of order 4 whose square transforms every operator of this cyclic group into its inverse. It should also be noted that when $p-1$ is divisible by 4 we can construct a group of order $16p$ which involves the dihedral group of order $4p$ as its operators which are squares thereunder.

The fact that the quaternion group is not composed of the squares of the operators of some other group is a special case of the following theorem: *There is no group which has the property that the squares of its operators constitute a dicyclic group.* A proof of this theorem may be made to depend upon the theorem that every group can be represented as a regular permutation group and hence if a group in which the squares of the operators constitute a dicyclic group did exist it would contain an operator of order 8 whose square would be a non-invariant operator of order 4 in the corre-

sponding dicyclic subgroup written as a simple isomorphism between two regular groups. This is impossible since one of these regular groups involves only two operators which are commutative with each of its operators.[2] A necessary and sufficient condition that the alternating group of degree n is composed of the squares of the operators of the corresponding symmetric group is that $n < 6$.

[1] G. A. Miller, these PROCEEDINGS, **19**, 848 (1933).

[2] *Ibid.*, **14**, 819 (1928).

GENERAL THEOREMS ON TRAJECTORIES AND LINES OF FORCE

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We study in this paper certain geometrical aspects of the motion of a particle in a general field of force. Most of the results have been stated briefly in previous abstracts.[1] We now wish to give the complete theorems and their proofs.

The theorems deal with the relations between the lines of force and the trajectories of any field. The particular situations considered are those in which a particle starts from rest or else is projected in the direction of the force with arbitrary speed. In these cases, our theorems give a first approximation in the path of the particle.

If a particle is placed at rest in a field of force, it begins to move along the line of force on which it is situated. However, on account of its inertia, it does not actually follow this line of force, but travels in a somewhat straighter path. That is, the trajectory and the line of force will agree as to initial direction, but differ in initial curvature, the trajectory obviously being situated between the common tangent line and the line of force. Our main result is as follows:

FUNDAMENTAL THEOREM I: *The curvature of the trajectory obtained by starting a particle from rest in any field of force is one-third the curvature of the line of force through the given point.*

As an illustration consider the magnetic field arising from the flow of electricity in a straight wire. The lines of force in a plane perpendicular to the wire are concentric circles. Our theorem states that an elementary pole placed in this plane will begin to move along a circle whose radius is three times the distance of the magnet from the wire (the actual trajectory is a spiral). Or consider the fall of a stone to the earth considered as an

ellipsoid. The path will be normal to the equipotential surfaces only at the start, and its curvature will be one-third that of the curve orthogonal to the equipotential surfaces. (Only when the line of force is straight will the trajectory coincide with the line of force.)

Theorem I is always valid, but is significant only if the curvature of the line of force is not zero at the given point. If the curvature vanishes, as for example, at a point of inflection, the same is true of the trajectory; and a separate discussion is necessary, involving third or higher derivatives. In this case we study the ratio of the infinitesimal departures of the path and the line of force from the common tangent line. In general this ratio would be 1 : 3, as stated, in effect, in Theorem I, but in the special case of zero curvature it is found to be 1 : 5, 1 : 7, 1 : 9, etc., depending on the order of contact of the line of force with the tangent line. *So the ratio always involves an odd integer.*

THEOREM II: *If the line of force has contact of n^{th} order with the tangent line, the trajectory produced by starting a particle from rest will also have contact of n^{th} order; and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent will be* $1 : 2n + 1$.

We may express this result intrinsically by introducing rates of variation of curvature with respect to arc length. Let γ_1 denote curvature and s_1 arc of the line of force; and γ_2 and s_2 the corresponding quantities for the trajectory.

THEOREM II′: *The ratio in the previous theorem can be written*[4]

$$\frac{d^{n-1}\gamma_2}{ds_2^{n-1}} : \frac{d^{n-1}\gamma_1}{ds_1^{n-1}} = 1 : 2n + 1.$$

We now consider the case in which a particle is projected with a non-zero velocity in the direction of the force. The particle will remain infinitesimally close to the tangent line to a higher order than when it is merely placed at rest in the field. As we increase the initial speed, the particle deviates from the tangent line less and less. This situation is described precisely in

THEOREM III: *If a particle is projected in the direction of the force with a speed different from zero, the initial curvature will be zero; and the infinitesimal departure from the common tangent will vary inversely as the square of the speed. That is* $d\gamma/ds$ *varies as* v^{-2}.

The initial curvature may be considered as a function of the speed of projection. From I and III it follows that this function is discontinuous when the initial velocity is zero, changing suddenly from zero to one-third the curvature of the line of force. A similar discontinuity occurs for higher contact between the line of force and the tangent line.

THEOREM IV: *The single infinity of paths obtained by starting at a given*

point in the force direction with varying speed under the conditions of Theorem II will have contact of order $n + 1$ with the common tangent; and will give departures from the common tangent varying inversely as the square of the speed[5]*; except for the single path due to zero speed for which case the contact will be of n^{th} order and the departure ratio will be of the form $1 : 2n + 1$.*

Theorem I remains true even if the motion takes place in a resisting medium, provided the resistance R_0 due to zero speed vanishes, as for example, in the case of a particle moving in a gas. However, we have in contrast

THEOREM V: *If R_0 does not vanish, as in the case of sliding friction, the ratio of the initial curvatures of the trajectory and the line of force, for a particle starting from rest, is $1:3 + 2\,R_0/F$ where F is the acting force at the position where the particle starts from rest.*

This obviously reduces to the ratio $1 : 3$ of our Fundamental Theorem I when R_0 vanishes.

All of these theorems can be generalized to apply to any differentiable positional field of force in flat or curved spaces of any dimensionality. The ratio $1:3$ remains fundamental.

2. First we give a synthetic proof of Theorem I.[2]

Let f be the acting force at the point where the particle is placed at rest in the field. We may take the mass to be unity. We choose for the x-axis the direction of f, that is, the tangent to the line of force, and for the y-axis the normal.

To a first approximation we may assume the x-force to remain constantly equal to f throughout the initial motion. We then approximate the y-force acting on the particle at any abscissa x by using for it the y-force that would act if the particle were at the same x on the initial circle of curvature of the line of force. By drawing a differential triangle we determine this y-force as xf/r where r is the initial radius of curvature of the line of force. The calculation of the path is now easy.

We have $\ddot{x} = f$; hence $x = {}^1/_2 ft^2$, since the initial velocity is zero. Then $\ddot{y} = xf/r = {}^1/_2\, f^2t^2/r$, so that $y = f^2t^4/24r$. Eliminating t we find $y = x^2/6r$. This parabolic approximation to the actual trajectory has for initial radius of curvature $3r$, which proves Theorem I.

A more intrinsic proof is indicated in Princeton Colloquium Lectures (1913) p. 9 and 104. The result is also included in the property given in *Trans. Amer. Math. Soc.* (1906), p. 416.

3. We now give analytic proofs for Theorems II and IV, of which I and III, respectively, are special cases. We omit proof of V, as a complete proof is given in my Colloquium, p. 104. The discussion there given applies also to Theorem I, and is actually a complete proof of that theorem.

As before, we choose the initial point as origin of coördinates and the tangent to the line of force as the x-axis; we assume unit mass, and we

write f for the force at the origin. Then by the hypothesis of Theorem II the equation of the line of force becomes of the form

$$y = ax^{n+1} + \ldots, (a \neq 0). \tag{1}$$

The components of the field are of the form[3]

$$\varphi(x, y) = f + a_{10}x + a_0, y + \ldots, \tag{2_1}$$

$$\psi(x, y) = b_{10}x + b_0, y + \ldots. \tag{2_2}$$

Let b_{m0} be the first non-vanishing coefficient of a power of x in (2_2). By definition of a line of force, (1) must satisfy the equation

$$y' = \psi(x, y)/\varphi\ (x, y). \tag{3}$$

Substituting from (1) and (2), we find

$$a(n+1)x^n + \ldots = \frac{b_{m0}}{f}x^m + \frac{b_{01}}{f}(ax^{n+1} + \ldots) + \ldots. \tag{4}$$

Equating lowest powers, $m = n$, and $b_{m0} = a(n+1)f$, and we have

$$\psi/\varphi = a(n+1)x^n + b_{01}y/f. \tag{5}$$

Now the trajectory is a solution of

$$\dot{x} = \varphi(x, y), \qquad \ddot{y} = \psi\ (x, y). \tag{6}$$

Since the initial velocity is zero, the parametric equations of the trajectory in terms of the time are

$$x = {}^1/_2 ft^2 + \ldots, \tag{7_1}$$

$$y = ct^p + \ldots. \tag{7_2}$$

To determine p and c we first calculate ψ/φ along the trajectory by means of the relations (6):

$$\psi/\varphi = \ddot{y}/\ddot{x} = \frac{cp(p-1)}{f}t^{p-2} + \ldots. \tag{8_1}$$

We find another form for ψ/φ by substituting (7) into (5):

$$\psi/\varphi = a(n+1)\left(\frac{f}{2}\right)^n t^{2n} + \ldots + \frac{b_{01}c}{f}t^p + \ldots. \tag{8_2}$$

The lowest power in (8_1) is t^{p-2}. But the only term in (8_2) which can

furnish this power is the one t^n. Hence $2n = p - 2$ and $cp(p-1)/f = a(n+1)(f/2)^n$, or $c = a(f/2)^{n+2}/(2n+1)$. Thus (7_2) becomes

$$y = \frac{a}{2n+1}(\tfrac{1}{2}ft^2)^{n+1} + \ldots . \qquad (9)$$

From (9) and (7_1),

$$y = \frac{a}{2n+1}x^{n+1} + \ldots . \qquad (10)$$

This is the cartesian equation of the path. It shows that the trajectory also has contact of the n^{th} order with the tangent line. The ratio of the infinitesimal departures of trajectory and line of force from the common tangent is given by the ratio of the leading terms of (10) and (1). This ratio is $1:2n+1$, which completes the proof of II.

4. To treat Theorem IV we repeat the above discussion up to equations (7). These, however, we must change to

$$x = vt + \tfrac{1}{2}ft^2 + \ldots , \qquad (7_1')$$

$$y = kt^q + \ldots , \qquad (7_2')$$

since the particle is projected in the direction of the force with an initial velocity v. As before, we determine q and k by equating the expressions for ψ/φ derived by substituting (7′) into (5) and (6):

$$\frac{kq(q-1)}{f}t^{q-2} + \ldots = a(n+1)v^nt^n + \ldots + \frac{b_{01}k}{f}t^q + \ldots . \qquad (8')$$

The lowest power on the left is t^{q-2}. The only term on the right that can furnish this power is the one in t^n. Hence $n = q - 2$ and $kq(q-1)/f = a(n+1)v^n$, or $k = afv^n/(n+2)$. Then $(7_2')$ becomes

$$y = \frac{af}{(n+2)v^2}(vt)^{n+2} + \ldots . \qquad (9')$$

From (9′) and $(7_1')$,

$$y = \frac{af}{(n+2)v^2}x^{n+2} + \ldots . \qquad (10')$$

Therefore the trajectory has contact of order $n+1$ with the tangent line. The ratio of the departures for two projectiles of different initial speeds is given by the ratio of their first terms in (10′), or what is the same, by the inverse ratio of the squares of their speeds, as stated in Theorem IV.

5. We remark in conclusion that all our results may be stated as analytic or geometric relationships between the second order differential equations

$$\frac{d^2x}{dt^2} = \varphi(x, y), \quad \frac{d^2y}{dt^2} = \psi(x, y) \tag{11}$$

and the first order equations

$$\frac{dx}{dt} = \varphi(x, y), \quad \frac{dy}{dt} = \psi(x, y). \tag{12}$$

Here (11) may be regarded as defining the totality of motions of a particle in a positional field (trajectories), and (12) as defining the flow of a liquid (stream lines). The latter (12) leads to a group of transformations, while the former (11) does not lead to a group.

The ∞^3 trajectories obey the fundamental equation

$$(\psi - y'\varphi)y''' = [\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2]y'' - 3\varphi y''^2 \tag{11}$$

first given in my paper in *Trans. Amer. Math. Soc.* (1906) p. 403, from which all the geometric properties may be derived.

The ∞^1 stream lines, or lines of force, obey the simple equation

$$\varphi y' - \psi = 0. \tag{12}$$

It was by comparing these two equations that I first obtained (1906) Theorem I. The other theorems of the present paper may be obtained in a similar fashion, but the calculations would seem to be cumbersome.

Theorem I remains valid in space of any dimensionality. It therefore applies, for example, to the fall of a stone to the earth considered as an ellipsoid; and to the problem of three bodies, of course using a representation by means of a single particle in higher space. Extension can be made to any riemannian space.

[1] *Trans. Amer. Math. Soc.* (1906–1910); *Bull. Amer. Math. Soc.*, 172 (1910); Princeton Colloquium Lectures, Differential Geometric Aspects of Dynamics, 1913, p. 9; *Science*, June, 671 (1932); *Zurich Congress Proc.*, **2**, 180 (1932). An application of Theorem I is given by Roever, *Bull. Amer. Math. Soc.*, 456 (1915). An experimental verification of the ratio 1:3 was carried out under my direction by H. Terrill, Columbia University, about fifteen years ago. The experimental verification of the ratios 1:5, 1:7, etc., would involve delicate technique.

[2] The synthetic proof of Theorem I was given by G. Comenetz, and the analytic proofs of Theorems II and IV were given by A. Fialkow, in my seminar.

[3] In the above proof we assume that the components of the field are analytic. Another proof by mathematical induction has been given by E. B. Callahan which assumes merely differentiability of sufficiently high order. Contact of infinite order for non-analytic curves will be discussed later by A. Fialkow.

[4] It can be shown that this *ratio of the curvature derivatives is always a projective invariant for any two curves having contact of n^{th} order.* For $n = 1$ (or simple contact) this reduces to a well-known theorem of Mehmke. In our dynamical problem we might expect a result of projective character because, if we apply Appell's classic transformation to a field of force, both the simple infinity of lines of force and the triple infinity

of trajectories are covariant. See Chapter III of my Princeton Lectures (edition of 1913 is out of print, but a new edition is now in the press).

[5] *It follows that* $\frac{d^n\gamma}{ds^n}$ *varies as* v^{-2}. (The derivatives of lower than nth order vanish). This remains true even for $n = 0$, that is, when the direction of initial velocity is different from the force direction, so that trajectories are not in contact with the line of force We then have the familiar result that the curvature varies inversely as the square of the speed. The next simplest case is $n = 1$. If we project a particle in the direction of the force, then the trajectories have, in general, ordinary inflexions, and *the quantity* $J = \frac{d\gamma}{ds}$ *(which I have termed inflexure) varies as* v^{-2}.

ON THE METRICAL TRANSITIVITY OF THE GEODESICS ON A SURFACE OF CONSTANT NEGATIVE CURVATURE

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The region under consideration is the upper half-plane. Let the differential element of arc length be defined as

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}, \quad y > 0.$$

The geodesics are then semicircles with centers on the real axis and the curvature is the negative constant -1. A certain Fuchsian group is assumed given, the fundamental region, R, of which is bounded by $4p$ arcs of geodesics (p an integer greater than one), the sides of R being congruent in pairs and the vertices constituting a single cycle, the interior angle at each vertex being $\pi/2p$. If congruent points are considered identical, there is thereby defined a closed orientable surface of genus p and of constant negative curvature.

Let $P(x, y)$ be a point of R, and let ϕ be an angular coördinate, ϕ measured in the positive direction from a direction parallel to the positive x-axis, $-\pi < \phi \leqq \pi$. Let M denote the three-dimensional region (x, y, ϕ) thus determined. A point of R and a direction at this point determines a point of M, hence any directed geodesic determines a point set of M, namely, those points of M determined by points and directions in R which are congruent to some point and direction of the given geodesic. Let G denote the set of points of M determined by a set g of geodesics. It is assumed that the set g is invariant under transformations of the given group. From a conjecture of Birkhoff,[1] if G is a measurable set, it should

be either of measure zero or of the measure of the region M. The object of this paper is to outline a proof that this is the case.

The work of J. Nielsen[2] has been of considerable use in obtaining this result, and a knowledge of the symbols used by Nielsen and their properties is assumed. Additional properties, not considered by Nielsen, must be derived. The Fuchsian group considered here has the real axis as principal circle, rather than the unit circle, but it is assumed that the fundamental region here considered can be obtained from that of Nielsen by a linear fractional transformation carrying the unit circle into the real axis, and the methods of Nielsen carry over without difficulty.

The first part of the proof consists of transforming the problem in three-dimensional sets into a problem in two-dimensional sets by the following procedure. Intervals δ_1 and δ_2 of the real axis are determined in a particular way and linear coördinates u and v, measured from the left endpoints of δ_1 and δ_2, respectively, are set up. A point of δ_1 paired with a point of δ_2 then determines a point of the (u, v) plane in the rectangle Δ which lies in the first quadrant with one vertex at the origin and sides of lengths δ_1 and δ_2, respectively, on the u and v axes. Thus those geodesics of the set g with initial point in δ_1 and terminal point in δ_2 determine a set E of the rectangle Δ, or to the set G in M there corresponds a set E in Δ.

THEOREM I. *If G is measurable and of positive measure, E is measurable and of positive measure.*

In the proof of this theorem we first consider the tubular region T of M determined by the family of directed geodesics with initial point in δ_1 and terminal point in δ_2, the points of T being determined by the segments of the geodesics of this family intercepted by R. If G is measurable the same is true of $G \cdot T$ (the points of G in T). Almost all of the sets S_e, the points of $G \cdot T$ in the plane $x = e$, are measurable, hence e can be chosen arbitrarily near zero. It is easily proved that the transformation from the set S_e to the set E is analytic, with non-vanishing Jacobian, and if S_e is measurable and of positive measure, E is measurable and of positive measure; if S_e is of measure zero, E is of measure zero. Hence E is measurable. If E is of measure zero, all of the sections S_e are of measure zero and $G \cdot T$ is of measure zero. But it can be assumed that the set g is made up of transitive geodesics, for it is known that the points of M determined by the intransitive geodesics constitute a set of zero measure.[3] But then every geodesic in the set g is congruent to a geodesic with initial point in δ_1 and terminal point in δ_2, and hence can be obtained from these last by a transformation of the group. The sum of a denumerable set of sets of zero measure is a set of zero measure and the stated theorem follows.

The second and most difficult part of the proof consists of proving that the set E being of positive measure it is of the measure of the rectangle Δ. The set has certain invariant properties due to the fact that the original

set g of geodesics is assumed invariant under a transformation of the group. If δ_3 and δ_4 are subintervals of δ_1 and δ_2, respectively, these determine a rectangle Δ_{34} of Δ. If under a transformation of the group δ_3 is transformed into δ'_3, a subinterval of δ_1, and δ_4 is transformed into δ'_4, a subinterval of δ_2, δ'_3 and δ'_4 determine a rectangle Δ'_{34} of Δ, and a transformation of Δ_{34} into Δ'_{34} is defined. The points of E in Δ_{34} must thereby be transformed into the set of points of E in Δ'_{34}.

The intervals δ_1 and δ_2 having been chosen as intervals of a certain stage,[4] they are divided into subintervals by the process of Nielsen, and these subintervals determine a net of rectangles in the rectangle Δ. It is here, in determining just how these rectangles change under the transformations of the group, that the technique of Nielsen is so useful.

It is impossible to give the details of the proof here, but the essential idea in the proof is to show that the part of E in any rectangle $\bar{\Delta}$ of the net in Δ is of measure greater than a certain fraction of the measure of $\bar{\Delta}$ itself, the fraction not depending on $\bar{\Delta}$. This is done by showing that a certain set of vertical strips in $\bar{\Delta}$ can, by transformations of the group, be transformed into a set of strips which are horizontal, of length δ_1 and stretch across the rectangle Δ. By properly choosing the original strips and the transformations, the resulting horizontal strips can be made arbitrarily numerous and vertically they are in a certain sense distributed regularly. It can be proved that the set of horizontal strips, if sufficiently numerous, contains a part of E of measure greater than a fixed constant times the total area of the strips. If the area and measure are not changed too much by the transformations used something can be said about the measure of the part of E in the original rectangle $\bar{\Delta}$. The following lemmas may indicate the direction of the proof.

LEMMA 1. *Let $\bar{\Delta}$ be an arbitrary rectangle of the net in Δ. A certain set σ of subrectangles of $\bar{\Delta}$ can be chosen which has certain properties to be stated, and which is always such that*

$$\mu(\sigma) > K_1 \mu(\bar{\Delta}),$$

where μ denotes measure and K_1 is a positive constant which does not depend on $\bar{\Delta}$.

LEMMA 2. *By transformations of the group the set σ can be transformed into a second set σ' of the rectangles of the net in Δ, and there exists a constant $K_2 > 0$ such that*

$$\mu(E \cdot \sigma) \geqq K_2 \mu(E \cdot \sigma').$$

LEMMA 3. *There exists a positive constant K_3 such that*

$$\mu(\sigma') > K_3 \mu(\sigma).$$

LEMMA 4. *There exists a positive constant K_4 such that if the set σ is chosen properly*

$$\mu(E \cdot \sigma') \geqq K_4\mu(\sigma')\mu(E).$$

Combining these results we have

$$\mu(E \cdot \bar{\Delta}) \geqq \mu(E \cdot \sigma) \geqq K\mu(\bar{\Delta}), \tag{1}$$

where $$K = K_1K_2K_3K_4\mu(E) > 0.$$

The constant K does not depend on the choice of $\bar{\Delta}$.

Now if E is of measure less than that of Δ, the complementary set $C(E)$ must be of positive measure and at the points of a set of positive measure the metric density of the set $C(E)$ must be 1. Thus there are points at which the metric density of E vanishes. But it is seen from (1) that the lower metric density of the set E cannot be less than K at any point and we have a contradiction. Thus we have the desired theorem.

THEOREM II. *The set E is of the measure of Δ.*

With the aid of this theorem, the remaining part of the proof is comparatively simple. We wish to prove that G is of the measure of M, or, what amounts to the same thing, that $C(G)$ is of measure zero. If $C(G)$ were of positive measure, by the reasoning used in the proof of Theorem I, $C(E)$ would be of positive measure and Theorem II would be contradicted.

THEOREM III. *The set G is of the measure of M.*

Measure has been used here in the sense in which it is usually defined in a Euclidean space. The volume of a cube is simply the product of its edges, where length is the usual Euclidean length. The procedure is perhaps open to objection because in defining the phase-space M, a metric should be introduced such that the distance between points made identical or near-by by congruence, should vanish or be small, as the case may be. A change in metric would change the volumes of the cubes considered in defining measure, but it seems evident that whatever metric is introduced, provided sufficient conditions as to analyticity are imposed, a set of zero measure would still be a set of zero measure and the validity of the results stated here would not be affected. The points where difficulty arises are the boundary points of M, but these constitute a set of zero measure and can be disregarded.

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[1] G. D. Birkhoff, "Probability and Physical Systems," *Bull. Am. Math. Soc.*, **38**, 371 (1932).

[2] J. Nielsen, "Untersuchungen zur Topologie der geschlossenen zewiseitigen Flächen," *Acta Mathematica*, **50**, 191–230 (1927).

[3] P. J. Myrberg, "Ein Approximationssatz für die Fuchsschen Gruppen," *Acta Mathematica*, **57**, 389–409 (1931).

G. A. Hedlund, "On the Measure of the Non-Special Geodesics on a Surface of Constant Negative Curvature," *Proc. Nat. Acad. Sci.*, **19**, 345–348 (1933).

The paper of Myrberg essentially contains the desired result. The author obtained the result without knowing of the paper by Myrberg. The method is different but it has not seemed desirable to publish the second proof in detail.

[4] Hedlund, loc. cit., p. 346.

ON THE THEORY OF LAPLACE INTEGRALS. II

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1. For notation and terminology we refer to our first note with the same title, quoted as (L1) in the sequel.[1]

Let $A(u)$ be of bounded variation on every finite closed interval $[0, \omega]$, let $A(0) = 0$, and form the integral

$$f(z) \sim \int_0^\infty e^{-zu}\, dA(u). \tag{1}$$

Denote the abscissas of simple, uniform and absolute convergence by $\sigma_0[f]$, $\sigma_u[f]$ and $\sigma_a[f]$, respectively. We have, of course, $-\infty \leqq \sigma_0 \leqq \sigma_u \leqq \sigma_a \leqq \infty$.

Suppose that $f(z)$ is holomorphic for $\Re(z) = x > \rho$, but not for $x > \rho - \epsilon$, $\epsilon > 0$. It is well known that a Laplace integral may be associated with an analytic function $f(z)$ without being convergent for any z. Even if $\sigma_0[f] < \infty$, we have ordinarily $\rho < \sigma_0[f]$. We have consequently the problem of evaluating a formal Laplace integral outside of its domain of convergence. This is an old problem, and numerous methods of summation are known which will effect the analytic continuation of $f(z)$ in some portion of its domain of existence—at least in the case in which $A(u)$ is a step function. In (L1) we considered the effectiveness of a method suggested by and essentially equivalent to the typical means of M. Riesz. We shall make some further remarks on this method in the present note, but our main object is to show that the multiplication theory of Laplace integrals is the common source of a large number of powerful methods of summing such integrals.

2. In this paragraph we shall assume that $f(z)$ is defined by (1) and $\sigma_0[f] < \infty$. Let

$$g(z) = \int_0^\infty e^{-zu}\, dB(u), \tag{2}$$

where $\sigma_0[g] < \infty$. Form

$$h(z) \sim \int_0^\infty e^{-zu}\, dC(u), \tag{3}$$

where

$$C(u) = \int_0^u A(u-v)\, dB(v) = \int_0^u B(u-v)\, dA(v), \tag{4}$$

the last two integrals being taken in the sense of Young-Stieltjes. The formal integral (3) need not converge, but an integration by parts gives[2]

$$h(z) = z \int_0^\infty e^{-zu}\, C(u)\, du \equiv f(z)\, g(z), \tag{5}$$

where the integral converges for $x > \max\{\sigma_0[f], \sigma_0[g]\}$. It follows that if (3) converges, its value is $f(z)\, g(z)$. This analogy with the theory of multiplication of infinite series goes much further. We can prove the following

THEOREM I. *If $\sigma_a[g] < \infty$, then (3) has a half-plane of convergence, and*

$$\sigma_0[h] \leqq \max\{\sigma_0[f], \sigma_a[g]\}, \tag{6}$$
$$\sigma_u[h] \leqq \max\{\sigma_u[f], \sigma_a[g]\}, \tag{7}$$
$$\sigma_a[h] \leqq \max\{\sigma_a[f], \sigma_a[g]\}. \tag{8}$$

This theorem is scarcely new, but we are not aware of its occurring in the literature in such complete form and under such general assumptions.

3. Theorem I serves as the point of departure of our theory of summability. A function $g(z)$ will be called a *multiplier* if there exists a half-plane $x > -\beta$ in which it is holomorphic, different from zero and representable by an absolutely convergent Laplace integral. The identity $f(z) = [g(z)]^{-1}\, g(z)\, f(z)$ yields the representation

$$f(z) = [g(z)]^{-1} \int_0^\infty e^{-zu}\, dC(u), \tag{9}$$

where $C(u)$ is given by (4). Let us denote the various abscissas of convergence of the integral in (9) by $\sigma_*[f; g]$, where the asterisk represents any one of the letters 0, u and a. From Theorem 1 we conclude

THEOREM 2. *If $\sigma_0[f] > -\beta$, then*

$$\sigma_*[f; g] \leqq \sigma_*[f]. \tag{10}$$

Thus the transformation by the multiplier $g(z)$ preserves convergence and limits. An application of the transformation may merely improve the convergence properties of a given Laplace integral. For instance, it may replace a simply convergent integral by one which is absolutely convergent. Of main interest, however, is the case in which $\sigma_*[f; g] < \sigma_*[f]$. An a

priori determination of the convergence abscissas $\sigma_*[f; g]$ with the aid of the analytic properties of $f(z)$ and $g(z)$ is seldom possible unless the multiplier $g(z)$ is strongly convergence inducing, in which case the three abscissas are liable to coincide. We can always obtain an upper limit of $\sigma_a[f; g]$ with the aid of Theorem I of our paper "On Moment Functions."[3] In the terminology of (L1) we get

THEOREM 3. *If* $f(z)\ g(z) \subset H_p(\sigma)$ *for some fixed* p, $1 \leqq p \leqq 2$, *then* $\sigma_a[f; g] \leqq \sigma$.

4. There are a number of interesting particular cases of this transformation. The case $g(z) = z^{-\alpha}$, $\Re(\alpha) > 0$, figured prominently in (L1). We can get greater flexibility by using $g(z) = (z + \beta)^{-\alpha}$ instead. Theorem 2 of (L1) shows the range of applicability of this multiplier; it is effective for sufficiently large positive α and β as long as $f(z)$ is of finite order with respect to the ordinate y. In order to bring out the fine structure of the properties of $f(z)$ we can employ the various functions of Hausdorff's logarithmic scale as multipliers.

If it is desired to get beyond the part of the plane where $f(z)$ is of finite order in y, we have to employ stronger means. The functions

$$\exp\left\{ -a\ [\log (z + \beta)]^{\gamma}\right\} \text{ and } \exp\left\{ -b\ (z + \beta)^{\alpha}\right\}, \tag{11}$$

where $a > 0$, $b > 0$, $\gamma > 1$, $1 > \alpha > 0$, belong to $H_2(-\beta + \epsilon)$ in the notation of (L1), and are consequently acceptable multipliers. For sufficiently large positive β the former function is effective in the half-plane where

$$\overline{\lim_{|y| \to \infty}} [\log |y|\ ]^{-\gamma} \log |f(x + iy)| < a, \tag{12}$$

the latter when

$$\overline{\lim_{|y| \to \infty}} |y|^{-\alpha} \log |f(x + iy)| < b. \tag{13}$$

We can permit $\alpha = 1$ in (11), but the corresponding transformation is useless for our purposes. We cannot construct with our present methods a multiplier $g(z)$ such that we are permitted to take $\alpha \geqq 1$ in (13). Indeed, such a function $g(z)$ must be bounded in a half-plane $x \geqq x_0$ and $|g(x_0 + iy)| < C \exp[-b|y|^{\alpha}]$, $\alpha \geqq 1$. It is a well-known consequence of the Phragmén-Lindelöf theorem that no such function $\not\equiv 0$ can exist.[4]

5. There exists, however, a transformation due to Hardy[5] which applies to Dirichlet series in the half-plane where

$$\overline{\lim_{|y| \to \infty}} |y|^{-1} \log |f(x + iy)| < \frac{\pi}{2\nu}. \tag{14}$$

This transformation is based upon the multiplier $\Gamma\left(\frac{z + \beta}{\nu}\right)$ which is

representable by an absolutely convergent Laplace integral where, however, the limits of integration are $-\infty$ and ∞ instead of 0 and ∞. Some other functions representable by such integrals, as, for instance, e^{z^2}, can be used for the same purpose. But a satisfactory theory of transformations based upon the use of such multipliers involves a systematic theory of multiplication of doubly infinite Laplace integrals which we intend to take up for discussion in a later note.

6. Let us return to the case $g(z) = (z + \beta)^{-\alpha}$, $\alpha > 0$, β real. For the sake of simplicity we write

$$\sigma_*[f(z);\ (z+\beta)^{-\alpha}] \equiv \sigma_*[f;\ \alpha, \beta]. \tag{15}$$

We have the following

THEOREM 4. *The three functions* $\sigma_*[f;\ \alpha,\ \beta]$ *are independent of* β *for sufficiently large values of* β *when* α *is fixed.* $\sigma_*[f;\ \alpha] = \lim_{\beta \to \infty} \sigma_*[f;\ a,\ \beta]$ *are monotone decreasing, convex functions of* α.

The first part of the theorem follows from Theorem 1 if we observe that $(z + \beta_1)^{-\alpha}\ (z + \beta_2)^{\alpha}$ is an acceptable multiplier in the half-plane $x > \max\ (-\beta_1, -\beta_2)$. For the case of simple convergence the second part is due to M. Riesz;[6] the remaining cases require some other ideas in addition, but the proof must be omitted here.

In the case of the Riemann Zeta function, or rather for $Z(s) = [1 - 2^{1-s}]\ \zeta(s)$, we obtain

$$\sigma_0[Z;\ \alpha] = -\alpha,\ \sigma_u[Z;\ \alpha] = \frac{1}{2} - \alpha,\ \sigma_a[Z;\ \alpha] = 1 - \alpha. \tag{16}$$

The first result is well known. In the first and the third equality $\alpha \geqq 0$, in the second $\alpha \geqq \frac{1}{2}$. The determination of $\sigma_u[Z;\ \alpha]$ for $0 < \alpha < \frac{1}{2}$ involves the customary difficulties encountered in the critical strip. Using the estimates, due to Hardy and Littlewood,[7] of $\mu[Z;\ \sigma]$ in the strip, we can get estimates of $\sigma_u[Z;\ a]$ for $0 < \alpha < \frac{1}{2}$. If $\alpha \geqq \frac{1}{2}$ is fixed, our results imply that the Laplace integral of $(s + \beta)^{-\alpha}\ Z(s)$ converges uniformly in the largest half-plane where this product is bounded. This is also true for $\alpha = 0$, but we do not know if it is true for $0 < \alpha < \frac{1}{2}$. All we can say in this case is that the convergence is uniform in the largest half-plane where the estimates mentioned above indicate that the product is bounded; but that is a different story. In case $\sigma_u[Z;\ \alpha] = \frac{1}{2} - \alpha$ also for $0 < \alpha < \frac{1}{2}$, Lindelöf's hypothesis would be proved.

[1] These PROCEEDINGS, 19, 908–912 (1933).

[2] D. V. Widder, "A Generalization of Dirichlet's Series and of Laplace's Integrals by Means of a Stieltjes Integral," *Trans. Amer. Math. Soc.*, 31, 694–743, esp. p. 715 (1929).

[3] These PROCEEDINGS, 19, 902–908 (1933).

[4] See, e.g., Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, I, p. 148, problem 327.

[5] G. H. Hardy, "The Application of Abel's Method of Summation to Dirichlet's Series," *Quart. Journal*, 47, 176–192 (1916).

[6] M. Riesz, "Sur un théorème de la moyenne et ses applications," *Acta Litt. ac Scient., Szeged*, 1, 114–126 (1922–23).

[7] See, e.g., E. Landau, *Vorlesungen über Zahlentheorie*, II, p. 55.